Sustav jednadžbi s generaliziranom vertikalnom koordinatom

Vježbe iz Dinamičke meteorologije II

Sustav jednadžbi s generaliziranom vertikalnom koordinatom

- pp. generalizirana vertikalna koordinata ζ je monotona funkcija visine z
- transformacija jednadžbi u koordinatni sustav $(x, y, \zeta, t) \rightarrow \quad z = z(x, y, \zeta, t)$
- bilo koja varijabla $A = A(x, y, z, t)$ može se opisati kao funkcija novih koordinata oblika:

$$A = A(x, y, z(x, y, \zeta, t), t)$$

- parcijalne derivacije

$$\frac{\partial A}{\partial \zeta} = \frac{\partial z}{\partial \zeta} \frac{\partial A}{\partial z}$$

$$\frac{\partial A}{\partial z} = \frac{\partial \zeta}{\partial z} \frac{\partial A}{\partial \zeta}$$

Sustav jednadžbi s generaliziranom vertikalnom koordinatom

- $s = (x, y, t)$

$$\left(\frac{\partial A}{\partial s}\right)_\zeta = \left(\frac{\partial A}{\partial s}\right)_z + \frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial s}\right)_\zeta$$

$$\left(\frac{\partial A}{\partial s}\right)_\zeta = \left(\frac{\partial A}{\partial s}\right)_z + \frac{\partial A}{\partial \zeta}\frac{\partial \zeta}{\partial z}\left(\frac{\partial z}{\partial s}\right)_\zeta$$

- horizontalni gradijent skalara A

$$\nabla_z A = \nabla_\zeta A - \frac{\partial A}{\partial \zeta}\frac{\partial \zeta}{\partial z}\nabla_\zeta z$$

- horizontalna divergencija vektora $\vec{v}$

$$\nabla_z \cdot \vec{v} = \nabla_\zeta \cdot \vec{v} - \frac{\partial \vec{v}}{\partial \zeta}\frac{\partial \zeta}{\partial z} \cdot \nabla_\zeta\, z$$

Sustav jednadžbi s generaliziranom vertikalnom koordinatom

- vertikalna brzina

$$w = \frac{dz}{dt} = \left(\frac{\partial z}{\partial t}\right)_\zeta + \left(\frac{\partial z}{\partial x}\right)_\zeta \frac{dx}{dt} + \left(\frac{\partial z}{\partial y}\right)_\zeta \frac{dy}{dt} + \frac{\partial z}{\partial \zeta}\frac{d\zeta}{dt}$$

$$w = \left(\frac{\partial z}{\partial t}\right)_\zeta + \vec{v} \cdot \nabla_\zeta z + \dot{\zeta}\frac{\partial z}{\partial \zeta}$$

- potpuni diferencijal

$$\frac{dA}{dt} = \left(\frac{\partial A}{\partial t}\right)_z + \vec{v} \cdot \nabla_z A + w\frac{\partial A}{\partial z}$$

$$\frac{dA}{dt} = \left(\frac{\partial A}{\partial t}\right)_\zeta + \vec{v} \cdot \nabla_\zeta A + \left[w - \left(\frac{\partial z}{\partial t}\right)_\zeta - \vec{v} \cdot \nabla_\zeta z\right]\frac{\partial \zeta}{\partial z}\frac{\partial A}{\partial \zeta}$$

$$\frac{dA}{dt} = \left(\frac{\partial A}{\partial t}\right)_\zeta + \vec{v} \cdot \nabla_\zeta A + \dot{\zeta}\frac{\partial A}{\partial \zeta}$$

Sustav jednadžbi s generaliziranom vertikalnom koordinatom

- **horizontalna komponenta jednadžbe gibanja**

$$\frac{d\vec{v}}{dt} = -\alpha \nabla_z p - f\vec{k} \times \vec{v}$$

$$\nabla_z p = \nabla_\zeta p - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial z}\nabla_\zeta z$$

- hidrostatička ravnoteža $\frac{\partial p}{\partial z} = -\rho g$

$$\nabla_z p = \nabla_\zeta p + \rho g \nabla_\zeta z = \nabla_\zeta p + \rho \nabla_\zeta \phi$$

$$\frac{d\vec{v}}{dt} = -\alpha \nabla_\zeta p - \nabla_\zeta \phi - f\vec{k} \times \vec{v} \quad (1)$$

Sustav jednadžbi s generaliziranom vertikalnom koordinatom

- **hidrostatička ravnoteža**

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial z} = -\rho g$$

- množenje izraza s $\alpha \frac{\partial z}{\partial \zeta}$

$$\alpha \frac{\partial p}{\partial \zeta} = -g\frac{\partial z}{\partial \zeta} = -\frac{\partial}{\partial \zeta}(gz) = -\frac{\partial \phi}{\partial \zeta}$$

$$\alpha \frac{\partial p}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} = 0 \quad (2)$$

Sustav jednadžbi s generaliziranom vertikalnom koordinatom

- **jednadžba kontinuiteta**

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{dt} + \nabla \cdot \vec{v} = 0$$

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{dt} = \frac{d}{dt}(\ln \rho) \qquad \nabla \cdot \vec{v} = \nabla_z \cdot \vec{v} + \frac{\partial w}{\partial z} = 0$$

$$\frac{d}{dt}(\ln \rho) + \nabla_\zeta \cdot \vec{v} + \frac{\partial \zeta}{\partial z}\frac{\partial w}{\partial \zeta} - \frac{\partial \zeta}{\partial z}\frac{\partial \vec{v}}{\partial \zeta} \cdot \nabla_\zeta z = 0$$

$$\frac{\partial w}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left[\left(\frac{\partial z}{\partial t}\right)_\zeta + \vec{v} \cdot \nabla_\zeta z + \dot{\zeta}\frac{\partial z}{\partial \zeta}\right]$$

$$\frac{\partial w}{\partial \zeta} = \left(\frac{\partial}{\partial t}\right)_\zeta \left(\frac{\partial z}{\partial \zeta}\right) + \vec{v} \cdot \nabla_\zeta \left(\frac{\partial z}{\partial \zeta}\right) + \dot{\zeta}\frac{\partial}{\partial \zeta}\left(\frac{\partial z}{\partial \zeta}\right) + \frac{\partial \vec{v}}{\partial \zeta} \cdot \nabla_\zeta z + \frac{\partial \dot{\zeta}}{\partial \zeta}\frac{\partial z}{\partial \zeta}$$

$$\frac{\partial w}{\partial \zeta} = \frac{d}{dt}\left(\frac{\partial z}{\partial \zeta}\right) + \frac{\partial \vec{v}}{\partial \zeta} \cdot \nabla_\zeta z + \frac{\partial \dot{\zeta}}{\partial \zeta}\frac{\partial z}{\partial \zeta}$$

Sustav jednadžbi s generaliziranom vertikalnom koordinatom

- **jednadžba kontinuiteta**

$$\frac{d}{dt}(\ln \rho) + \frac{d}{dt}\ln\left(\frac{\partial z}{\partial \zeta}\right) + \nabla_\zeta \cdot \vec{v} + \frac{\partial \dot{\zeta}}{\partial \zeta} = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{1}{\alpha}\frac{\partial z}{\partial \zeta}\right)\right] + \nabla_\zeta \cdot \vec{v} + \frac{\partial \dot{\zeta}}{\partial \zeta} = 0$$

- koristimo izraz za hidrostatičku ravnotežu

$$\frac{d}{dt}\left[\ln\left(\frac{1}{g}\left|\frac{\partial p}{\partial \zeta}\right|\right)\right] + \nabla_\zeta \cdot \vec{v} + \frac{\partial \dot{\zeta}}{\partial \zeta} = 0 \quad (3)$$

Sustav jednadžbi s generaliziranom vertikalnom koordinatom

- 1. stavak termodinamike

$$\frac{đQ}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt}$$

$$\frac{đQ}{dt} = c_p \left[\left(\frac{\partial T}{\partial t} \right)_\zeta + \vec{v} \cdot \nabla_\zeta T + \dot{\zeta} \frac{\partial T}{\partial \zeta} \right] - \alpha \left[\left(\frac{\partial p}{\partial t} \right)_\zeta + \vec{v} \cdot \nabla_\zeta p + \dot{\zeta} \frac{\partial p}{\partial \zeta} \right]$$

- potencijalna temperatura $\vartheta = T \left(\frac{p_0}{p} \right)^{\frac{R}{c_p}}$ / ln, $\frac{d}{dt}$

$$c_p T \frac{d}{dt} (ln\vartheta) = -\frac{RT}{p} \frac{dp}{dt} + c_p \frac{dT}{dt}$$

$$\left(\frac{đQ}{dt} \right)_\zeta = c_p T \left(\frac{d}{dt} (\ln \vartheta) \right)_\zeta \quad (4)$$

Sustav jednadžbi s generaliziranom vertikalnom koordinatom

$$\frac{d\vec{v}}{dt} = -\alpha \nabla_\zeta p - \nabla_\zeta \phi - f\vec{k} \times \vec{v}$$

$$\alpha \frac{\partial p}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} = 0$$

$$\frac{d}{dt}\left[\ln\left(\frac{1}{g}\left|\frac{\partial p}{\partial \zeta}\right|\right)\right] + \nabla_\zeta \cdot \vec{v} + \frac{\partial \dot{\zeta}}{\partial \zeta} = 0$$

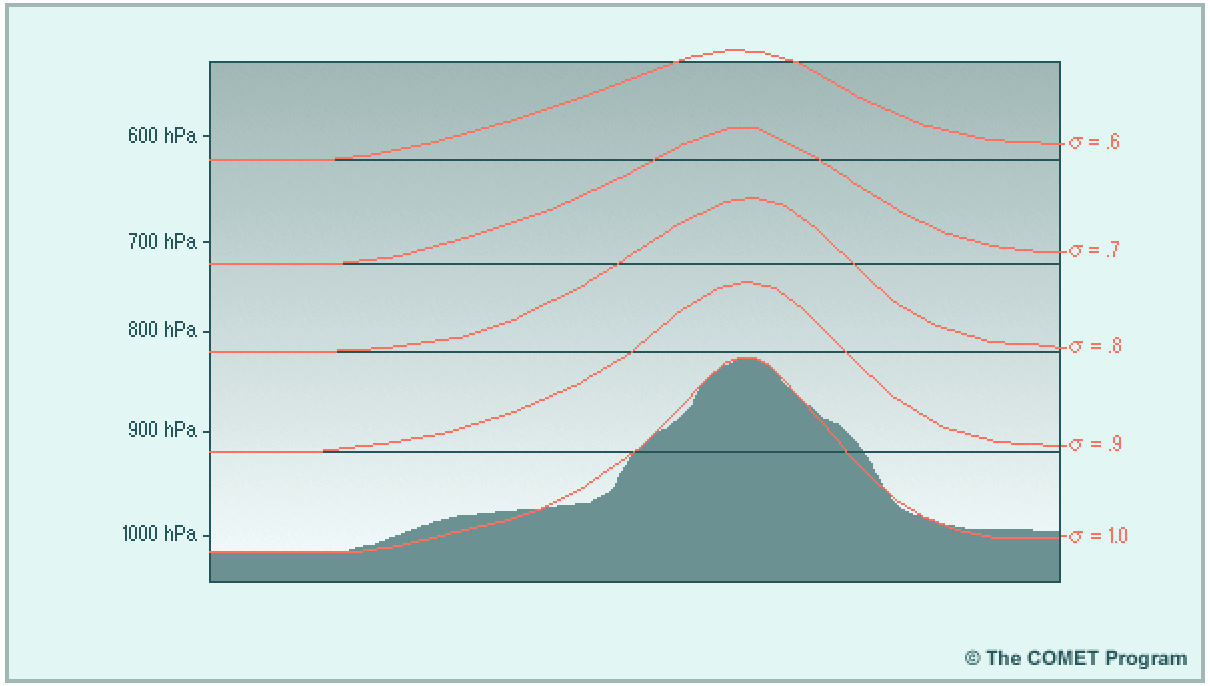
$$\left(\frac{đQ}{dt}\right)_\zeta = c_p T \left(\frac{d}{dt}(\ln \vartheta)\right)_\zeta$$

Sustavi s različitim vertikalnim koordinatama

- Kartezijev (pravokutni) KS (x,y,z)
- izobarni KS (x,y,p)
- adijabatički ili izentropski KS (x,y,ϑ)
- izotermni KS (x,y,T)
- sigma KS (x,y,σ) → uvažava orografiju

$$\sigma = \frac{p}{p_s}$$

gdje je p_s - surface pressure



Literatura

- Geoffrey K. Vallis, Atmospheric and oceanic fluid dynamics (Second Edition), Cambridge University Press, 2017, Chapter 2 – Effects of Roatiation and Stratification , Pages 79-82
- James R. Holton, Gregory J. Hakim, An Introduction to Dynamic Meteorology (Fifth Edition), Academic Press, 2013, Chapter 1 - Introduction, Pages 23-24, https://doi.org/10.1016/B978-0-12-384866-6.00001-5
- https://ocw.mit.edu/courses/earth-atmospheric-and-planetary-sciences/12-950-atmospheric-and-oceanic-modeling-spring-2004/lecture-notes/lec25.pdf
- https://barnes.atmos.colostate.edu/COURSES/AT601_F15/lecture_material/07_lecture_notes_handouts.pdf

Primjeri i zadatci

1. Iz osnovnih jednadžbi u sustavu s generaliziranom vertikalnom koordinatom, izvedite osnovne jednadžbe u:
    a) (x,y,p) koordinatnom sustavu
    b) (x,y,ϑ) koordinatnom sustavu

2. Iz horizontalne komponente jednadžbe gibanja u sustavu s generaliziranom vertikalnom koordinatom, izvedite izraz za geostrofički vjetar u:
    a) (x,y,z) koordinatnom sustavu
    b) (x,y,p) koordinatnom sustavu
    c) (x,y,ϑ) koordinatnom sustavu
    d) (x,y,σ) koordinatnom sustavu
    e) (x,y,T) koordinatnom sustavu

Rješenja

1. Iz osnovnih jednadžbi u sustavu s generaliziranom vertikalnom koordinatom, izvedite osnovne jednadžbe u:

a) (x,y,p) koordinatnom sustavu

$$\frac{d\vec{v}}{dt} = -\alpha \nabla_p p - f\vec{k} \times \vec{v} - \nabla_p \phi \;\; \rightarrow \;\; \frac{d\vec{v}}{dt} = -f\vec{k} \times \vec{v} - \nabla_p \phi$$

$$\alpha \frac{\partial p}{\partial p} + \frac{\partial \phi}{\partial p} = 0 \;\; \rightarrow \;\; \frac{\partial \phi}{\partial p} = -\alpha$$

$$\frac{d}{dt}\left(\ln \frac{1}{g}\frac{\partial p}{\partial p}\right) + \nabla_p \cdot \vec{v} + \frac{\partial \dot{p}}{\partial p} = 0 \;\; \rightarrow \;\; \nabla_p \cdot \vec{v} + \frac{\partial}{\partial p}\frac{\partial p}{\partial t} = 0 \;\; \rightarrow \;\; \frac{\partial \omega}{\partial p} = -\nabla_p \cdot \vec{v}$$

gdje je vertikalna brzina $\omega = \frac{\partial p}{\partial t}$

$$\left(\frac{đQ}{dt}\right)_p = c_p T \left(\frac{d}{dt}(\ln \vartheta)\right)_p = c_p T \left(\frac{d}{dt}\ln T - \frac{R}{c_p}\frac{d}{dt}\ln p\right) = c_p T \left(\frac{1}{T}\frac{dT}{dt} - \frac{R}{c_p}\frac{1}{p}\frac{dp}{dt}\right) = c_p \frac{dT}{dt} - \alpha\omega$$

1. Iz osnovnih jednadžbi u sustavu s generaliziranom vertikalnom koordinatom, izvedite osnovne jednadžbe u:

   b) (x,y,ϑ) koordinatnom sustavu

$$\frac{d\vec{v}}{dt} = -\alpha \nabla_\vartheta p - f\vec{k} \times \vec{v} - \nabla_\vartheta \phi$$

$$đq = c_p dT - \alpha dp \ \ uz\ dq = 0 \ \rightarrow \ c_p dT = \alpha dp$$

$$\frac{d\vec{v}}{dt} = -\nabla_\vartheta c_p T - f\vec{k} \times \vec{v} - \nabla_\vartheta (gz) = -\nabla_\vartheta \left(c_p T + gz\right) - f\vec{k} \times \vec{v}$$

gdje je $M = c_p T + gz$ Montgomeryjev potencijal

$$\alpha \frac{\partial p}{\partial \vartheta} + \frac{\partial \phi}{\partial \vartheta} = 0 \ \ \rightarrow c_p \frac{\partial T}{\partial \vartheta} + \frac{\partial (gz)}{\partial \vartheta} = 0$$

$$\frac{d}{dt}\left(\ln \frac{1}{g} \frac{\partial p}{\partial \vartheta}\right) + \nabla_\vartheta \cdot \vec{v} + \frac{\partial \dot{\vartheta}}{\partial \vartheta} = 0$$

$$\left(\frac{đQ}{dt}\right)_\vartheta = c_p T \left(\frac{d}{dt} (\ln \vartheta)\right)_\vartheta$$

2. Iz jednadžbe gibanja u sustavu s generaliziranom vertikalnom koordinatom, izvedite izraz za geostrofički vjetar u:

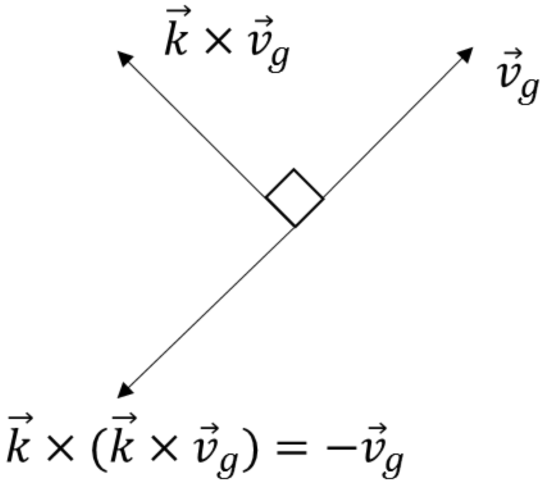
$$\frac{d\vec{v}}{dt} = -\alpha \nabla_\zeta p - \nabla_\zeta \phi - f\vec{k} \times \vec{v}$$

geostrofički vjetar pripada u kategoriju stacionarnih / ravnotežnih gibanja → $\frac{d\vec{v}}{dt} = 0$

$$f\vec{k} \times \vec{v}_g = -\alpha \nabla_\zeta p - \nabla_\zeta \phi$$

$$\vec{k} \times (\vec{k} \times \vec{v}_g) = -\frac{1}{f}\alpha \vec{k} \times \nabla_\zeta p - \frac{1}{f}\vec{k} \times \nabla_\zeta \phi$$

$$\vec{v}_g = \frac{\alpha}{f}\vec{k} \times \nabla_\zeta p + \frac{1}{f}\vec{k} \times \nabla_\zeta \phi$$



a) (x,y,z) koordinatnom sustavu

$$\vec{v}_g = \frac{\alpha}{f}\vec{k} \times \nabla_z p + \frac{1}{f}\vec{k} \times \nabla_z \phi \;\leftarrow \nabla_z \phi = \nabla_z (gz) = 0$$

$$\vec{v}_g = \frac{\alpha}{f}\vec{k} \times \nabla_z p$$

2. Iz jednadžbe gibanja u sustavu s generaliziranom vertikalnom koordinatom, izvedite izraz za geostrofički vjetar u:

b) (x,y,p) koordinatnom sustavu

$$\vec{v}_g = \frac{\alpha}{f}\vec{k} \times \nabla_p p + \frac{1}{f}\vec{k} \times \nabla_p \phi$$

$$\vec{v}_g = \frac{1}{f}\vec{k} \times \nabla_p (gz)$$

c) (x,y,σ) koordinatnom sustavu

$$\sigma = \frac{p}{p_s} \rightarrow p = \sigma p_s \quad i \quad \frac{\partial \phi}{\partial p} = -\alpha$$

$$\alpha \nabla_\sigma p = -\frac{\partial \phi}{\partial p}\nabla_\sigma p = -\frac{\partial \phi}{\partial (\sigma p_s)}\nabla_\sigma (\sigma p_s) = -\frac{1}{p_s}\frac{\partial \phi}{\partial \sigma}\nabla_\sigma (\sigma p_s) = -\sigma \frac{\partial \phi}{\partial \sigma}\nabla_\sigma (\ln p_s)$$

$$\vec{v}_g = -\frac{1}{\mathrm{f}}\vec{k} \times \left(\sigma \frac{\partial \phi}{\partial \sigma}\nabla_\sigma \ln p_s\right) + \frac{1}{f}\vec{k} \times \nabla_\sigma \phi$$

d) (x,y,T) koordinatnom sustavu

$$\vec{v}_g = \frac{\alpha}{f}\vec{k} \times \nabla_T p + \frac{1}{f}\vec{k} \times \nabla_T \phi = \frac{RT}{fp}\vec{k} \times \nabla_T p + \frac{1}{f}\vec{k} \times \nabla_T \phi$$

$$\vec{v}_g = \frac{1}{f}\vec{k} \times \nabla_T (RT \ln p + \phi)$$

2. Iz jednadžbe gibanja u sustavu s generaliziranom vertikalnom koordinatom, izvedite izraz za geostrofički vjetar u:

e) (x,y,ϑ) koordinatnom sustavu

$$\vec{v}_g = \frac{\alpha}{f}\vec{k} \times \nabla_\vartheta p + \frac{1}{f}\vec{k} \times \nabla_\vartheta \phi$$

$$đq = c_p dT - \alpha dp \;\; uz \; dq = 0 \;\rightarrow\; c_p dT = \alpha dp$$

$$\vec{v}_g = \frac{\alpha}{f}\vec{k} \times \frac{c_p}{\alpha}\nabla_\vartheta T + \frac{1}{f}\vec{k} \times \nabla_\vartheta (gz)$$

- potencijalna temperatura $\vartheta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}$ / ln, $\nabla_\vartheta \cdot$

$$\ln\vartheta = \ln T + \frac{R}{c_p}\ln p_0 - \frac{R}{c_p}\ln p$$

$$0 = \frac{1}{T}\nabla_\vartheta \mathrm{T} - \frac{\mathrm{R}}{\mathrm{c_p}}\frac{1}{\mathrm{p}}\nabla_\vartheta p$$

- jednadžba stanja idealnog plina $p = \rho RT$

$$0 = \frac{1}{T}\nabla_\vartheta \mathrm{T} - \frac{\mathrm{R}}{\mathrm{c_p}}\frac{1}{\rho RT}\nabla_\vartheta p$$

2. Iz jednadžbe gibanja u sustavu s generaliziranom vertikalnom koordinatom, izvedite izraz za geostrofički vjetar u:

e) (x,y,ϑ) koordinatnom sustavu

$$0 = \frac{1}{T}\nabla_\vartheta \mathrm{T} - \frac{\mathrm{R}}{\mathrm{c_p}}\frac{1}{\rho RT}\nabla_\vartheta p$$

- množenje s T i sređivanje oblika daje:

$$\nabla_\vartheta T = \frac{\alpha}{c_p}\nabla_\vartheta p \;\rightarrow \nabla_\vartheta p = \frac{c_p}{\alpha}\nabla_\vartheta T$$

$$\vec{v}_g = \frac{1}{f}\vec{k} \times \nabla_\vartheta\left(c_p T + gz\right) = \frac{1}{f}\vec{k} \times \nabla_\vartheta M$$