2015./2016.

5. V KDVP, $A \in L(V)$ $r(A) = k > 0$

Pokažite da postoji ortonormiran skup $\{e_1, e_2, \dots, e_k\}$ i vektori $f_1, f_2, \dots, f_k$ t.d.

$$Ax = \sum_{i=1}^{k} \langle x, f_i \rangle e_i, \quad \forall x \in V$$

Rj: Neka je $\{e_1, \dots, e_k\}$ ONB za $\operatorname{Im} A$. Za $x \in V$

$$Ax = \sum_{i=1}^{k} \lambda_i e_i$$ za neke $\lambda_i \in \mathbb{F}$. Odredimo λ_i:

$$\langle Ax, e_j \rangle = \sum_{i=1}^{k} \lambda_i \langle e_i, e_j \rangle = \lambda_j \qquad j \in \{1, \dots, k\}$$

$$\Rightarrow Ax = \sum_{i=1}^{k} \langle Ax, e_i \rangle e_i$$

$$= \sum_{i=1}^{k} \langle x, A^* e_i \rangle e_i$$

Stavimo $f_i = A^* e_i$, $i \in \{1, \dots, k\}$. Tada je

$$Ax = \sum_{i=1}^{k} \langle x, f_i \rangle e_i.$$ Uočite da to vrijedi $\forall x \in V$

($f_i = A^* e_i$ ne ovisi o x-u).

2016./2017.

1. (a) Dokažimo $\operatorname{Ker} A = \operatorname{Ker} A^* A$

$\boxed{\subseteq}$ Za $x \in \operatorname{Ker} A \Rightarrow Ax = 0 \Rightarrow A^* A x = 0$

$\Rightarrow x \in \operatorname{Ker} A^* A$

$\boxed{2}$ Za $x \in \mathrm{Ker}\, A^*A \Rightarrow A^*Ax = 0 \Rightarrow \langle A^*Ax, x\rangle = 0$

$\Rightarrow \langle Ax, Ax\rangle = 0 \Rightarrow Ax = 0 \Rightarrow x \in \mathrm{Ker}\, A$.

b) Kako je $\mathrm{Ker}\, A = \mathrm{Ker}\, A^*A \Rightarrow d(A) = d(A^*A)$

$n = \dim V$ Po teoremu o rangu i defektu

$r(A) = n - d(A)$

$r(A^*A) = n - d(A^*A) = n - d(A) = r(A)$

$\Rightarrow r(A^*A) = r(A)$

c) $V = \mathbb{R}^2$ (e) kanonska baza

$$A(e) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^*(e) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$A^*A(e) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} \mathrm{Im}\, A &= [\{e_1\}] \\ \mathrm{Im}\, A^*A &= [\{e_2\}] \end{aligned}\right\} \Rightarrow \mathrm{Im}\, A \neq \mathrm{Im}\, A^*A$$

$\boxed{2017./2018.}$ 3. $V = \left[\left\{ \underbrace{\begin{pmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A}, \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix}}_{B} \right\}\right]$

a) $G = V \oplus V^\perp$

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 \\ 0 & 0 & x_6 \end{pmatrix} \in V^\perp \Leftrightarrow X \perp A \ \&\ X \perp B$$

$\Leftrightarrow 2x_1 + x_2 - 2x_3 = 0 \ \&\ 4x_4 + 3x_5 = 0$

$\Leftrightarrow x_2 = 2x_3 - 2x_1 \ \&\ x_5 = -\frac{4}{3}x_4$

$$V^{\perp} = \left[\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&-2&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&2&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&-\frac{4}{3}\\0&0&0\end{pmatrix}\right\}\right]$$

$$= \left[\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&-\frac{4}{3}\\0&0&0\end{pmatrix}, \begin{pmatrix}1&-2&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&2&1\\0&0&0\\0&0&0\end{pmatrix}\right\}\right]$$

Ortonormirajte ovu bazu ...

b) $I = X + Y \quad , X \in V, Y \in V^{\perp}$

$X = I - Y$

$Y = \langle I, Y_1 \rangle Y_1 + \langle I, Y_2 \rangle Y_2 + \langle I, Y_3 \rangle Y_3 + \langle I, Y_4 \rangle Y_4,$

pri čemu je $\{Y_1, Y_2, Y_3, Y_4\}$ ONB za $V^{\perp}$ iz a) dijela.