$\boxed{2018/2019}$ **1.** $q(x,y,z) = 2x^2 + 2y^2 - z^2 + 2xy + 4xz - 4yz$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$\sigma(A) = \{-3, 3\}$$

$$V_A(-3) = \left[\left\{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right\}\right], \quad V_A(3) = \left[\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}\right]$$

Ortonormiranjem dobijemo $e_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $e_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$$b_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$

$$e_3 = \frac{\cancel{\frac{1}{5}}\begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}}{\cancel{\frac{1}{5}}\sqrt{30}} = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = Q^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dots$$

U koordinatama
$$\left.\begin{matrix} x_1 = \\ y_1 = \\ z_1 = \end{matrix}\right\} \text{izrazite ih preko } x, y, z$$

forma ima kanonski oblik

2018/2019

3. $$(p|q)=\int_{-1}^{1} p(x)q(x)\,dx$$

$$F(p)=\int_{-1}^{1} p(|x|)\,dx-p'(0)$$

Nađite $q\in\mathcal{P}_2$ t.d. $F(p)=(p|q)\quad\forall p\in\mathcal{P}_2$

Baza za $\mathcal{P}_2$ je $\{1,t,t^2\}$ $\quad e_1(t)=1,\ e_2(t)=t,\ e_3(t)=t^2$

$$F(p)=(p|q)\ \forall p\in\mathcal{P}_2 \iff \begin{aligned} &F(e_1)=(e_1|q) \quad (1)\\ &\&\\ &F(e_2)=(e_2|q) \quad (2)\\ &\&\\ &F(e_3)=(e_3|q) \quad (3)\end{aligned}$$

Stavimo oznaku $q(t)=at^2+bt+c$

$$(e_1|q)=\int_{-1}^{1} 1\cdot(at^2+bt+c)\,dt=\frac{a}{3}t^3\Big|_{-1}^{1}+\frac{b}{2}t^2\Big|_{-1}^{1}+ct\Big|_{-1}^{1}=$$

$$=\frac{2}{3}a+2c$$

$$F(e_1)=\int_{-1}^{1} e_1(|x|)\,dx-e_1'(0)=$$

$$=\int_{-1}^{1} dx-0=x\Big|_{-1}^{1}=2$$

Iz (1) $\Rightarrow$ $\frac{2}{3}a+2c=2$

Sada to isto napravite za (2) i (3) (izračunajte i sredite). Dobit ćete sustav tri jednadžbe s tri nepoznanice a, b i c. Riješite sustav i dobit ćete polinom q.

$\boxed{2018/2019}$

(4.) b) $A, B \in L(U)$ $\quad B^*A = 0$

Treba dokazati $\ker(A+B) = \ker A \cap \ker B$

$\boxed{\supseteq}$ Za $x \in \ker A \cap \ker B \Rightarrow Ax = 0 \;\&\; Bx = 0$

$\Rightarrow (A+B)x = Ax + Bx = 0 + 0 = 0$

$\Rightarrow x \in \ker(A+B)$

$\boxed{\subseteq}$ Za $x \in \ker(A+B) \Rightarrow (A+B)x = 0$

$\Rightarrow Ax + Bx = 0 \Rightarrow Ax = -Bx$

$\Rightarrow B^*(Ax) = -B^*(Bx)$

$\Rightarrow \underbrace{(B^*A)}\, x = -B^*Bx$

$\Rightarrow 0 = -B^*Bx$

$\Rightarrow B^*Bx = 0$

$\Rightarrow \langle B^*Bx, x\rangle = \langle 0, x\rangle = 0$

$\Rightarrow \langle Bx, Bx\rangle = 0 \Rightarrow \|Bx\| = 0 \Rightarrow Bx = 0 \Rightarrow x \in \ker B$ (1)

$$\underbrace{Ax+Bx}_{0}=0 \Rightarrow Ax=0 \Rightarrow x\in \ker A \quad (2)$$

Iz (1) i (2) $\Rightarrow x \in \ker A \cap \ker B$

$\boxed{2017/2018}$ 4. b)

$$A=\begin{bmatrix} 5 & -2 & 2 \\ -2 & 2 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

$\sigma(A)=\{6,-3\}$

$$V_A(6)=\left\{\begin{bmatrix}2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\1\\0\end{bmatrix}\right\} \qquad v_1=\begin{bmatrix}2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

$$V_A(-3)=\left\{\begin{bmatrix}-1\\-2\\2\end{bmatrix}\right\}$$

Ortonormiranjem baze za $V_A(-3)$ i $V_A(6)$ dobijemo matricu U. No, uočite da je

$$V_A(6)=\left[\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}^{\to v_1+v_2},\begin{bmatrix}2\\0\\1\end{bmatrix}\right\}\right].$$

Ortonormiranjem te baze (i baze za $V_A(-3)$) dobijemo matricu V za koju je $A=VDV^{t}$. Prvi stupac matrice V nije proporcionalan niti jednom stupcu matrice U.

Uočite da takvih matrica V ima koliko god hoćete.

2016./2017. (popravni)

(3.) $a_1 = (i,0,0)$, $a_2 = (1,0,1)$, $a_3 = (0,i,i)$

$$f(a_1) = i, \quad f(a_2) = 1, \quad f(a_3) = 2$$

$$f(x) = \langle x, v \rangle \quad \forall x \in \mathbb{C}^3$$

Uočimo da je $\{a_1, a_2, a_3\}$ baza za $\mathbb{C}^3$.

$$f(x) = \langle x, v \rangle \ \forall x \in \mathbb{C}^3 \iff \left\{ \begin{array}{ll} f(a_1) = \langle a_1, v \rangle & (1) \\ \& & \\ f(a_2) = \langle a_2, v \rangle & (2) \\ \& & \\ f(a_3) = \langle a_3, v \rangle & (3) \end{array} \right.$$

Stavimo $v = (a,b,c)$

(1): $f(a_1) = \langle a_1, v \rangle = \langle (i,0,0), (a,b,c) \rangle = i \cdot \bar{a}$

$$\Rightarrow i = i \cdot \bar{a} \Rightarrow \bar{a} = 1 \Rightarrow \boxed{a = 1}$$

(2): $f(a_2) = \langle a_2, v \rangle = \langle (1,0,1), (a,b,c) \rangle =$

$$= 1 \cdot \bar{a} + 1 \cdot \bar{c} = 1 \cdot 1 + 1 \cdot \bar{c}$$

$$\Rightarrow 1 = 1 + 1 \cdot \bar{c} \Rightarrow \boxed{c = 0}$$

(3): $f(a_3) = \langle a_3, v \rangle = 0 \cdot \bar{a} + i \cdot \bar{b} + i \cdot \bar{c} = i \cdot \bar{b} \Rightarrow 2 = i \cdot \bar{b} \Rightarrow \bar{b} = -2i$

$$\Rightarrow \boxed{b = 2i}$$

$\boxed{2016/2017}$ (popravni)

5.

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$$

$$(Ap)(t) = t p'(t)$$

(a) $e_1(t) = 1$

$e_2(t) = t$

$e_3(t) = t^2$

Uputa: 1. Ortonormirajte tu bazu s obzirom na dani skalarni produkt. Dobijete bazu $(f) = \{f_1, f_2, f_3\}$.

2. Izračunajte matricu $A(f)$.

3. Kako je (f) ONB $\Rightarrow A^*(f) = (A(f))^*$.

4. Sada izračunajte $A^*(f_1)$ u bazi (f), $A^*(f_2)$ u bazi (f), $A^*(f_3)$ u bazi (f) (samo pročitajte iz matrice A^* koju dobijete pod 3.)

Zapamtite: $A^*(f) = (A(f))^*$ zato što je (f) ONB.

(Recimo $A^*(e) \neq (A(e))^*$ jer (e) nije ONB)

b) Ako postoji takav skalarni produkt, onda nađemo neku o s obzirom na taj skalarni produkt. Tada je $A(g)$ unitarna matrica pa je $\sigma(A) \subseteq \{-1, 1\}$. (vidjeti predavanja)

Međutim $(Ae_1)(t) = t \cdot 0 = 0$

$(Ae_2)(t) = t \cdot 1 = t$

$(Ae_3)(t) = t \cdot 2t = 2t^2$

$$A(e) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Rightarrow \sigma(A) = \{1, 2\}$

$\Rightarrow \sigma(A) \not\subseteq \{-1, 1\}$

$\Rightarrow$ ne postoji skalarni produkt uz koji je A unitaran operator.

$\boxed{2014./2015.}$ (popravni)

3. $\langle A, B \rangle = tr\left(A \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} B^T\right)$

$f(A) = uAu^T \quad u = [2 \ 3]$

$f(A) = \langle A, T \rangle \quad \forall A \in M_2(\mathbb{R})$

$f(A) = \langle A, T \rangle \ \forall A \in M_2(\mathbb{R}) \iff$

$f(E_i) = \langle E_i, T \rangle \quad \forall i \in \{1, 2, 3, 4\}$ $(*)$, gdje je

$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Stavimo $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Sada to uvrstite u $(*)$. Dobit ćete četiri lin. jednadžbe s nepoznanicama a, b, c, d. Riješite sustav i dobit ćete matricu T.