⑥ $B : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$

$$B(at^2+bt+c) = (a-c)t^3 + bt^2 - 3ct + 2a+b$$

$B(1) = -t^3 - 3t$ $\qquad (e) = \{1, t, t^2\}$

$B(t) = t^2 + 1$ $\qquad (f) = \{1, t, t^2, t^3\}$

$B(t^3) = t^3 + 2$

$$[B]_{(f,e)} = \begin{bmatrix} 0 & 1 & 2 \\ -3 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$(e') = \{1, 1-t^2, t+t^2\}$

$(f') = \{1+t+t^2, 2t, 1+t^3, -1-t\}$

$$[B]_{(f',e')} = [I]_{(f',f)} \cdot [B]_{(f,e)} \cdot [I]_{(e,e')} =$$

$$= \left([I]_{(f,f')}\right)^{-1} \cdot [B]_{(f,e)} \cdot [I]_{(e,e')}$$

$$[I]_{(e,e')} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad [I]_{(f,f')} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Uvrstite i sredite

Postoji li par baza u kojem B ima matrični prikaz

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} ?$$

Dakle, za $(f'') = \{-t^3-3t, t^2+1, t^3+2, 1\}$ je $[B]_{(f'',e)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Rj. nije jedinstveno

Ovo je lakše nego što mislite :)

$B(1) = -t^3 - 3t =: f_1'' = 1 \cdot f_1''$

$B(t) = t^2 + 1 =: f_2'' = 1 \cdot f_2''$

$B(t^3) = t^3 + 2 =: f_3'' = 1 \cdot f_3''$

$$[B]_{(f'',e)} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ B(e_1) & B(e_2) & B(e_3) \end{matrix} \end{array}$$

Za f_4'' možemo uzeti bilo što tako da (f'') bude baza, npr. $f_4'' = 1$

7. Zadan je linearan operator $A: \mathbb{R}^2 \to \mathbb{R}^2$ sa svojstvenim vrijednostima $\lambda_1 = -1$ i $\lambda_2 = 4$ i pripadnim sv. potprostorima $V_{\lambda_1}(A) = [\{(1,2)\}]$ i $V_{\lambda_2}(A) = [\{(0,1)\}]$. Odredite djelovanje operatora A na opći vektor $(x,y) \in \mathbb{R}^2$ tako da ga najprije zapišete u takvoj bazi da matrica bude dijagonalna, a zatim iskoristite matrice prijelaza.

Rj: $v_1 = (1,2)$, $v_2 = (0,1)$ $\quad (b) = \{v_1, v_2\}$ je jedna baza prostora $\mathbb{R}^2$.

$A(v_1) = -v_1$, $A(v_2) = 4v_2$ $\quad \Rightarrow [A]_{(b)} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

$$\Rightarrow [A]_{(e)} = [I]_{(e,b)} \cdot [A]_{(b)} \cdot [I]_{(b,e)}$$

$$[I]_{(e,b)} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$[I]_{(b,e)} = [I]_{(e,b)}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\Rightarrow [A]_{(e)} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 0 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -10 & 4 \end{bmatrix}$$

$$\Rightarrow [A(x,y)]_{(e)} = [A]_{(e)} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -10 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -10x+4y \end{bmatrix}$$

$$\Rightarrow \boxed{A(x,y) = (-x, -10x+4y)}$$

⑨ Za sljedeće matrice A odredite matricu P t.d. $A = PDP^{-1}$, gdje je D dijagonalna matrica (ako je to moguće):

a) $A = \begin{bmatrix} 2 & -7 \\ -3 & -2 \end{bmatrix}$ $\quad k_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -7 \\ -3 & -2-\lambda \end{vmatrix} =$

$$= -(2-\lambda)(2+\lambda) - 21 = -(4-\lambda^2) - 21 = \lambda^2 - 25 = (\lambda-5)(\lambda+5)$$

$\sigma(A) = \{-5, 5\}$ $\quad a(-5) = a(5) = 1$ $\quad \Rightarrow 1 \leq g(-5) \leq a(-5) = 1 \Rightarrow g(-5) = 1$

$\quad 1 \leq g(5) \leq a(5) = 1 \quad \Rightarrow a(5) = 1$

Dakle, $a(-5) = g(-5)$, $\quad a(5) = g(5)$, $\quad a(-5) + a(5) = 2 = \dim \mathbb{R}^2$

$\Rightarrow$ A je dijagonalizabilna.

- $V_A(-5) = \operatorname{Ker}(A + 5I)$ $\quad A + 5I = \begin{bmatrix} 7 & -7 \\ -3 & 3 \end{bmatrix}$ $\quad (A+5I) \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff$

$$\left.\begin{aligned} 7x_1 - 7x_2 &= 0 \\ -3x_1 + 3x_2 &= 0 \end{aligned}\right\} \iff x_1 = x_2$$

$V_A(-5) = [\{(1,1)\}]$

- $V_A(5) = \operatorname{Ker}(A - 5I)$ $\quad A - 5I = \begin{bmatrix} -3 & -7 \\ -3 & -7 \end{bmatrix}$ $\quad (A-5I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff$

$$-3x_1 - 7x_2 = 0 \iff$$

$$x_2 = -\frac{3}{7}x_1$$

$V_A(5) = [\{(1, -\frac{3}{7})\}]$

- $[A]_{(e)} = \underbrace{[I]_{(e,b)}}_{P} \cdot [A]_{(b)} \cdot \underbrace{[I]_{(b,e)}}_{P^{-1}}$ $\quad D = [A]_{(b)} = \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}$

$(b) = \{(1,1), (1, -\frac{3}{7})\}$ $\quad P = [I]_{(e,b)} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{3}{7} \end{bmatrix}$

Uočimo, $V_A(5) = [\{(-7, 3)\}]$ pa možemo staviti i $P = \begin{bmatrix} 1 & -7 \\ 1 & 3 \end{bmatrix}$

Matrica P nije jedinstveno određena.

(5) $A=\begin{bmatrix} 1 & -5 & 8 \\ 1 & -2 & 1 \\ 2 & -1 & -5 \end{bmatrix}$

$$k_A(\lambda)=\begin{vmatrix} 1-\lambda & -5 & 8 \\ 1 & -2-\lambda & 1 \\ 2 & -1 & -5-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -1 & -5-\lambda \end{vmatrix} - \begin{vmatrix} -5 & 8 \\ -1 & -5-\lambda \end{vmatrix} + 2\begin{vmatrix} -5 & 8 \\ -2-\lambda & 1 \end{vmatrix} =$$

$$= (1-\lambda)(\lambda^2+7\lambda+11) - (25+5\lambda+8) + 2(-5+16+8\lambda)$$

$$= (1-\lambda)(\lambda^2+7\lambda+11) + 11\lambda - 11 =$$

$$= (1-\lambda)(\lambda^2+7\lambda+11-11) = -(\lambda-1)\lambda(\lambda+7)$$

$\sigma(A)=\{0,1,-7\} \Rightarrow a(0)=a(1)=a(7)=1 \Rightarrow g(0)=g(1)=g(7)=1.$ $a(0)+a(1)+a(7)=3$

$\Rightarrow A$ je dijagonalizabilna

- $V_A(0) = \operatorname{Ker} A$ $\quad \begin{bmatrix} 1 & -5 & 8 \\ 1 & -2 & 1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow$

$$\begin{bmatrix} \boxed{1} & -5 & 8 \\ 1 & -2 & 1 \\ 2 & -1 & -5 \end{bmatrix} \begin{matrix} /\cdot(-1) \;\; /\cdot(-2) \\ \oplus \\ \oplus \end{matrix} \sim \begin{bmatrix} 1 & -5 & 8 \\ 0 & 3 & -7 \\ 0 & 9 & -21 \end{bmatrix} \begin{matrix} /\cdot(-3) \\ \oplus \end{matrix} \sim \begin{bmatrix} 1 & -5 & 8 \\ 0 & 3 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 5x_2 + 8x_3 = 0 \qquad 3x_2 - 7x_3 = 0$

$$x_2 = \frac{7}{3}x_3$$

$$x_1 = 5x_2 - 8x_3 = \frac{35}{3}x_3 - 8x_3 = \frac{11}{3}x_3$$

$V_A(0) = [\{(11, 7, 3)\}]$

- $V_A(1) = \operatorname{Ker}(A-I)$ $\quad A-I=\begin{bmatrix} 0 & -5 & 8 \\ \boxed{1} & -3 & 1 \\ 2 & -1 & -6 \end{bmatrix} \begin{matrix} \\ /\cdot(-2) \\ \oplus \end{matrix} \sim \begin{bmatrix} 0 & -5 & 8 \\ 1 & -3 & 1 \\ 0 & 5 & -8 \end{bmatrix} \begin{matrix} /\cdot(-1) \\ \\ \oplus \end{matrix} \sim \begin{bmatrix} 0 & -5 & 8 \\ 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$-5x_2 + 8x_3 = 0 \quad \Rightarrow x_2 = \frac{8}{5}x_3$

$x_1 - 3x_2 + x_3 = 0 \quad \Rightarrow x_1 = 3x_2 - x_3 = \frac{24}{5}x_3 - x_3 = \frac{19}{5}x_3$

$V_A(1) = [\{(19, 8, 5)\}]$

• $V_A(-7) = \mathrm{Ker}(A+7I) = ?$ $\qquad A+7I = \begin{bmatrix} 8 & -5 & 8 \\ \boxed{1} & 5 & 1 \\ 2 & -1 & 2 \end{bmatrix} \begin{matrix} /\cdot(-2) \ /\cdot(-8) \\ \\ \end{matrix}$

$$\sim \begin{bmatrix} 0 & -45 & 0 \\ 1 & 5 & 1 \\ 0 & -11 & 0 \end{bmatrix} \begin{matrix} /:(-45) \\ \\ /:(-11) \end{matrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} /\cdot(-1) \ /\cdot(-5) \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_2 = 0, \quad x_1 + x_3 = 0$$

$$V_A(7) = [\{(1,0,-1)\}]$$

$$[A]_{(e)} = \underbrace{[I]_{(e,b)}}_{P} \underbrace{[A]_{(b)}}_{D} \cdot \underbrace{[I]_{(b,e)}}_{P^{-1}} \qquad D = [A]_{(b)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$P = [I]_{(e,b)} = \begin{bmatrix} 11 & 19 & 1 \\ 7 & 8 & 0 \\ 3 & 5 & -1 \end{bmatrix}$$

c) $A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -3 & 0 & 3 \\ -1 & -3 & 1 & -1 \end{bmatrix} \qquad k_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 & 0 & 0 \\ -1 & -1-\lambda & 1 & 0 \\ -1 & -3 & -\lambda & 3 \\ -1 & -3 & 1 & -1-\lambda \end{vmatrix}$

$$= -3 \begin{vmatrix} -2-\lambda & 1 & 0 \\ -1 & -1-\lambda & 1 \\ -1 & -3 & 1 \end{vmatrix} - (1+\lambda) \begin{vmatrix} -2-\lambda & 1 & 0 \\ -1 & -1-\lambda & 1 \\ -1 & -3 & -\lambda \end{vmatrix}$$

$$= -3 \left(- \begin{vmatrix} -2-\lambda & 1 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} -2-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} \right) - (1+\lambda) \left(- \begin{vmatrix} -2-\lambda & 1 \\ -1 & -3 \end{vmatrix} - \lambda \begin{vmatrix} -2-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} \right)$$

$$= (3 + (1+\lambda)) \begin{vmatrix} -2-\lambda & 1 \\ -1 & -3 \end{vmatrix} + (-3 + \lambda(1+\lambda)) \begin{vmatrix} -2-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} =$$

$$= (4+\lambda)(3\lambda+7) + (\lambda^2+\lambda-3)(\lambda^2+3\lambda+2+1) = \lambda^4 + 4\lambda^3 + 6\lambda^2 + 13\lambda + 19$$

$\sigma(A) = \emptyset \Rightarrow$ A se ne može dijagonalizovati