6. zadaća, 1. d) $D: \mathbb{C}_{\mathbb{R}}^3 \to \mathbb{R}^n$

$$\dim(\mathbb{C}_{\mathbb{R}}^3) = 6, \quad \dim(\mathbb{R}^n) = n$$

- Ako je $n = 6$, onda je $\dim(\mathbb{C}_{\mathbb{R}}^3) = \dim(\mathbb{R}^6) = 6$, pa postoji izomorfizam $D: \mathbb{C}_{\mathbb{R}}^3 \to \mathbb{R}^6$. Stavimo

$$D(1,0,0) = (1,0,0,0,0,0)$$
$$D(0,1,0) = (0,1,0,0,0,0)$$
$$D(0,0,1) = (0,0,1,0,0,0)$$
$$D(i,0,0) = (0,0,0,1,0,0)$$
$$D(0,i,0) = (0,0,0,0,1,0)$$
$$D(0,0,i) = (0,0,0,0,0,1)$$

Proširimo D „po linearnosti"

$$D(x_1 + iy_1, x_2 + iy_2, x_3 + iy_3) =$$

$$= (x_1, x_2, x_3, y_1, y_2, y_3), \quad x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$$

D preslikava bazu prostora $\mathbb{C}_{\mathbb{R}}^3$ u bazu prostora $\mathbb{R}^6$ pa je D izomorfizam (ujedno je i epimorfizam i monomorfizam).

- Ako je $n < 6$: $\quad d(D) + r(D) = \dim(\mathbb{C}_{\mathbb{R}}^3) = 6$

$$r(D) \leq \dim \mathbb{R}^n < 6 \Rightarrow d(D) = 6 - r(D) > 6 - 6 = 0$$

$$\Rightarrow D \text{ ne može biti monomorfizam}$$

D može biti epimorfizam, npr. neka je $\{e_1, \ldots, e_n\}$ kanonska baza za $\mathbb{R}^n$. Definiramo D na bazi $\{\underbrace{(1,0,0)}_{a_1}, \underbrace{(0,1,0)}_{a_2}, \underbrace{(0,0,1)}_{a_3}, \underbrace{(i,0,0)}_{a_4}, \overbrace{(0,i,0)}^{a_5}, \underbrace{(0,0,i)}_{a_6}$

za $\mathbb{C}^3_{\mathbb{R}}$ t.d. $D(a_i)=e_i$ $i\in\{1,2,\dots,n\}$ $(n<6)$ i $D(a_i)=(0,0,0,0,0,0)$ (za $i>n$)

proširimo po linearnosti. Tada je $\{D(a_1),\dots,D(a_6)\}$ s.i. za $\operatorname{Im} D$

$\Rightarrow$ $\{e_1,\dots,e_n\}$ je s.i. za $\operatorname{Im} D$ $\Rightarrow$ $\operatorname{Im} D=\mathbb{R}^n$. $\Rightarrow$ D je epimorfizam.

Za $n<6$ D ne može biti izomorfizam jer je za $n<6$

$\dim(\mathbb{C}^3_{\mathbb{R}})\neq\dim(\mathbb{R}^n)$

- Ako je $n>6$: $d(D)+r(D)=6$ $\Rightarrow r(D)\leq 6<\dim\mathbb{R}^n$

  $\Rightarrow$ D ne može biti epimorfizam, a onda ne može biti ni izomorfizam.

D može biti monomorfizam. Stavimo npr.

$$D(a_i)=e_i \qquad i\in\{1,\dots,6\}$$

(i proširimo po linearnosti)

Tada je $r(D)=\dim\{e_1,\dots,e_6\}=6 \Rightarrow d(D)=\dim\mathbb{C}^3_{\mathbb{R}}-6=0$

$\Rightarrow$ d je monomorfizam

6. zadaća, 1. e) $E: \mathbb{C}_{\mathbb{R}}^3 \to S_3(\mathbb{R})$

$$S_3(\mathbb{R}) = \left\{ \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\} =$$

$$= \left[\left\{ E_{11}, E_{12}+E_{21}, E_{13}+E_{31}, E_{22}, E_{23}+E_{32}, E_{33} \right\} \right]$$

$\dim S_3(\mathbb{R}) = 6 = \dim \mathbb{C}_{\mathbb{R}}^3 \Rightarrow S_3(\mathbb{R})$ & $\mathbb{C}_{\mathbb{R}}^3$ su izomorfni

Definirajmo E npr. na ovaj način:

$$E(1,0,0) = E_{11}$$
$$E(0,1,0) = E_{12}+E_{21}$$
$$E(0,0,1) = E_{13}+E_{31}$$
$$E(i,0,0) = E_{22}$$
$$E(0,i,0) = E_{23}+E_{32}$$
$$E(0,0,i) = E_{33}$$

Proširimo po linearnosti: $E(x_1+iy_1, x_2+iy_2, x_3+iy_3) =$

$$= x_1 E_{11} + x_2 \cdot (E_{12}+E_{21}) + x_3 \cdot (E_{13}+E_{31}) +$$
$$y_1 E_{22} + y_2 (E_{23}+E_{32}) + y_3 E_{33} \qquad x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$$

E je izomorfizam jer preslikava bazu u bazu.

(ujedno je i monomorfizam i epimorfizam).

6. Zadaćk, 2. zad c) $C: M_2(\mathbb{R}) \to P_2(\mathbb{R})$,

$$C\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a-c+((\lambda+2)b+c-d)t+(b+c+\lambda d)t^2$$

$$C\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1, \quad C\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = (\lambda+2)t+t^2$$

$$C\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = -1+t+t^2, \quad C\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = -t+\lambda t^2$$

$\operatorname{Im} C$ razapeta vektorima $\{C(E_{11}), C(E_{12}), C(E_{21}), C(E_{22})\}$

$$= \{1, (\lambda+2)t+t^2, -1+t+t^2, -t+\lambda t^2\}$$

$$= \{1, -1+t+t^2, -t+\lambda t^2, (\lambda+2)t+t^2\}$$

$$-t+\lambda t^2 = \alpha\cdot 1+\beta(-1+t+t^2) = \alpha-\beta+\beta t+\beta t^2 \iff$$

$$\alpha-\beta=0, \quad \beta=-1, \quad \beta=\lambda \iff \alpha=\beta=\lambda=-1 \ (*)$$

1° slučaj: $\lambda=-1 \Rightarrow \operatorname{Im} C = [\{1, -1+t+t^2, -t-t^2, t+t^2\}]$

$$= [\{1, t+t^2\}]$$

$$\Rightarrow r(C)=2 \Rightarrow d(C)=\dim M_2(\mathbb{R})-2=2$$

(C nije epimorfizam)

$$C\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \iff \begin{matrix} a-c=0 \\ b+c-d=0 \\ b+c-d=0 \end{matrix} \qquad \begin{matrix} a=c \\ d=b+c \end{matrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \operatorname{Ker} C \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & b \\ c & b+c \end{bmatrix} = c\cdot\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\operatorname{Ker} C = \left[\left\{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}\right]$$

$\Rightarrow$ C nije monomorfizam, a onda ni izomorfizam.

2. slučaj: $\lambda \neq -1$ iz $(*)$ slijedi da je za $\lambda \neq -1$ skup

$\{1, -1+t+t^2, -t+\lambda t^2\}$ linearno nezavisan.

$$\Rightarrow \underbrace{[\{1, -1+t+t^2, -t+\lambda t^2\}]}_{\text{dimenzija } 3} = \mathcal{P}_2(\mathbb{R})$$

$$\Rightarrow [\{1, -1+t+t^2, -t+\lambda t^2, (\lambda+2)t+t^2\}] = \mathcal{P}_2(\mathbb{R})$$

$\Rightarrow \operatorname{Im} C = \mathcal{P}_2(\mathbb{R}) \Rightarrow r(C) = 3 = \dim \mathcal{P}_2(\mathbb{R}) \Rightarrow C$ je epimorfizam

$\Rightarrow d(C) = \dim M_2(\mathbb{R}) - 3 = 1 \Rightarrow C$ nije monomorfizam,

a onda ni izomorfizam

$$C\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \iff a = c$$

$$\left.\begin{aligned} (\lambda+2)b + c - d &= 0 \\ b + c + \lambda d &= 0 \end{aligned}\right| \ominus \quad (\lambda+1)b - (\lambda+1)d = 0$$

$$\underbrace{(\lambda+1)}_{\neq 0}(b-d) = 0 \qquad \boxed{b = d}$$

$$c = -b - \lambda b = -(1+\lambda)b$$

$$\operatorname{Ker} C = \left\{ \begin{bmatrix} -(1+\lambda)b & b \\ -(1+\lambda)b & b \end{bmatrix} \right\} = \left[\left\{ \begin{bmatrix} -(1+\lambda) & 1 \\ -(1+\lambda) & 1 \end{bmatrix} \right\}\right]$$