1

Derivacija

1.1 Tehnika deriviranja

Definicija. Neka je $f\colon I \to \mathbb{R}$ funkcija, $I \subseteq \mathbb{R}$ otvoreni interval i $c \in I$. Kažemo da je f **derivabilna** u točki c ako postoji

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

i taj limes ocnačavamo s $f'(c)$.

Još se koristi i Leibnizova oznaka $\frac{dy}{dx}$.

Zadatak 1.1 Koristeći definiciju, odredite derivaciju funkcija:

(a) $f(x) = \alpha = \text{const.}$

(b) $f(x) = x$

(c) $f(x) = \frac{1}{x}$

(d) $f(x) = \cos x$

Rješenje.

(a) $f'(c) = \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = \lim\limits_{x\to c} \dfrac{\alpha-\alpha}{x-c} = \lim\limits_{x\to c} \dfrac{0}{x-c} = 0$

(b) $f'(c) = \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = \lim\limits_{x\to c} \dfrac{x-c}{x-c} = \lim\limits_{x\to c} 1 = 1$

(c) $f'(c) = \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = \lim\limits_{x\to c} \dfrac{\frac{1}{x}-\frac{1}{c}}{x-c} = \lim\limits_{x\to c} \dfrac{-1}{cx} = -\dfrac{1}{c^2}$, $c \neq 0$

(d) $f'(c) = \lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim\limits_{x\to c} \frac{\cos x - \cos c}{x-c} = \lim\limits_{x\to c} \frac{-2\sin\frac{x+c}{2}\sin\frac{x-c}{2}}{x-c} = -\sin c$

△

Zadatak 1.2 Nađite primjer funkcije koja nije derivabilna.

Rješenje. Neka je $f(x) = |x|$. Limes

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|x|}{x},$$

ne postoji, jer se limesi slijeva i zdesna u 0 razlikuju:

$$\begin{aligned} \lim_{x\to 0-} \frac{|x|}{x} &= \lim_{x\to 0-} \frac{-x}{x} = -1, \\ \lim_{x\to 0+} \frac{|x|}{x} &= \lim_{x\to 0+} \frac{x}{x} = 1. \end{aligned}$$

△

Vrijede sljedeća pravila deriviranja:

$$\begin{aligned} (\alpha f + \beta g)' &= \alpha f' + \beta g' \\ (f\cdot g)' &= f'\cdot g + f\cdot g' \\ \left(\frac{f}{g}\right)' &= \frac{f'\cdot g - f\cdot g'}{g^2} \end{aligned}$$

U Leibnizovoj notaciji gornja pravila glase:

$$\begin{aligned} \frac{d}{dx}(\alpha u + \beta v) &= \alpha\frac{d\,u}{d\,x} + \beta\frac{d\,v}{d\,x} \\ \frac{d}{dx}(u\cdot v) &= \frac{d\,u}{d\,x}\,v + u\,\frac{d\,v}{d\,x} \\ \frac{d}{dx}(\frac{u}{v}) &= \frac{\frac{d\,u}{d\,x}\,v - u\,\frac{d\,v}{d\,x}}{v^2} \end{aligned}$$

Zadatak 1.3 Derivirajte sljedeće funkcije:

(a) $f(x) = x^5 - 4x^3 + 2x - 3$

(b) $f(x) = \frac{ax^6 + b}{\sqrt{a^2+b^2}},\ a, b \in \mathbb{R}, a^2 + b^2 \neq 0$

(c) $f(x) = 3\sqrt{x}$

(d) $f(x) = 3x^{\frac{2}{3}} - 2x^{\frac{5}{2}} + x^{-3}$

(e) $f(x) = \dfrac{2x+3}{x^2 - 5x + 5}$

(f) $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

(g) $f(x) = e^x \arcsin x$

(h) $f(x) = \operatorname{arctg} x + \operatorname{arcctg} x$

(i) $f(x) = 2^x \cdot x$

(j) $f(x) = \arcsin x \cdot \operatorname{Arsh} x$

Rješenje.

(a) $f'(x) = 5x^4 - 12x^3 + 2$

(b) $f'(x) = \dfrac{6ax^5}{\sqrt{a^2+b^2}}$

(c) $f'(x) = \dfrac{3}{2\sqrt{x}}$

(d) $f'(x) = 2x^{-1/3} - 5x^{3/2} - 3x^{-4}$

(e) $f'(x) = \dfrac{-2x^2 - 6x + 25}{(x^2 - 5x + 5)^2}$

(f) $f'(x) = \dfrac{-2}{(\sin x - \cos x)^2}$

(g) $f'(x) = e^x \, \arcsin x + \dfrac{e^x}{\sqrt{1-x^2}}$

(h) $f'(x) = 0$

(i) $f'(x) = 2^x \, x \ln 2 + 2^x$

(j) $f'(x) = \dfrac{\operatorname{Arsh} x}{\sqrt{1-x^2}} + \dfrac{\arcsin x}{\sqrt{1+x^2}}$

△

Derivacija kompozicije funkcija

Teorem. Neka su $I, J \subseteq \mathbb{R}$ otvoreni intervali i neka su $f\colon I \to \mathbb{R}$ i $g\colon J \to \mathbb{R}$ funkcije takve da je $f(I) \subseteq J$. Ako je f derivabilna u $c \in I$ i g derivabilna u $d = f(c) \in J$, onda je $g \circ f$ derivabilna u c i vrijedi

$$(g \circ f)'(c) = g'(d) \cdot f'(c).$$

■

U Lebnizovoj notaciji:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Primjer. Derivirajmo funkciju $f(x) = \left(\frac{2x+3}{5}\right)^8$.

Prikažimo f kao $f = g \circ h$, gdje je

$$\begin{aligned} h(x) &= \frac{2x+3}{5} \implies h'(x) = \frac{2}{5} \\ g(x) &= x^8 \implies g'(x) = 8x^7. \end{aligned}$$

Tada je

$$f'(x) = g'(h(x)) \cdot h'(x) = 8h(x)^7 \cdot \frac{2}{5} = \frac{16}{5}\left(\frac{2x+3}{5}\right)^7.$$

△

Primjer. Derivirajmo funkciju $f(x) = e^{\sin^2 x}$.

Prikažimo f kao $f = l \circ g \circ h$, gdje je

$$\begin{aligned} h(x) &= \sin x \implies h'(x) = \cos x \\ g(x) &= x^2 \implies g'(x) = 2x, \\ l(x) &= e^x \implies l'(x) = e^x, \end{aligned}$$

Tada je

$$\begin{aligned} f'(x) &= (l \circ (g \circ h))'(x) = l'((g \circ h)(x)) \cdot (h \circ l)'(x) \\ &= l'((g \circ h)(x)) \cdot h'(l(x)) \cdot l'(x) = e^{\sin^2 x} \cdot 2 \sin x \cdot \cos x = e^{\sin^2 x} \sin 2x. \end{aligned}$$

△

Primjer. Derivirajmo funkciju $f\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$, $f(x)=\ln|x|$.

Vrijedi

- $x>0 \implies f(x)=\ln x \implies f'(x)=\frac{1}{x}$,
- $x<0 \implies f(x)=\ln(-x) \implies f'(x)=\frac{1}{-x}\cdot(-1)=\frac{1}{x}$

pa je $f'(x)=\dfrac{1}{x}$.

Zadatak 1.4 Koristeći pravilo za derivaciju kompozicije funkcija (*chain rule*), derivirajte sljedeće funkcije:

(a) $f(x)=\sqrt{\operatorname{ctg} x}$

(b) $f(x)=\sqrt{1+\arcsin x}$

(c) $f(x)=\sqrt{xe^x+x^2}$

(d) $f(x)=\sqrt[3]{2e^x-2^x+1}+\ln^5 x$

(e) $f(x)=\dfrac{1}{5^{x^2}}$

(f) $f(x)=\log\sin x$

(g) $f(x)=\left(\frac{1+2x^{20}}{1-2x^{20}}\right)^{100}$

(h) $f(x)=\ln\ln\left(3-2x^2\right)$

(i) $f(x)=\arccos e^{x^2}$

(j) $f(x)=\dfrac{\sqrt{2x^2-2x+1}}{x}$

(k) $f(x)=\ln\left(\sqrt{1+e^x}-1\right)-\ln\left(\sqrt{1+e^x}+1\right)$

(l) $f(x)=\ln\dfrac{(x-2)^5}{(x+1)^3}$

Rješenje.

(a) $f'(x)=\dfrac{-1}{2\sin^2 x\sqrt{\operatorname{ctg} x}}$

(b) $f'(x)=\dfrac{1}{2\sqrt{(1-x^2)(1+\arcsin x)}}$

(c) $f'(x) = \dfrac{(x+1)e^x + 1}{2\sqrt{(x+1)e^x}}$

(d) $f'(x) = \dfrac{2e^x - 2^x \ln 2}{3\sqrt[3]{(2e^x - 2^x + 1)^2}} + \dfrac{5\ln^4 x}{x}$

(e) $f'(x) = -2x5^{-x^2} \ln 5$

(f) $f'(x) = \dfrac{\operatorname{tg} x}{\ln 10}$

(g) $f'(x) = 100 \left(\dfrac{1 + 2x^{20}}{1 - 2x^{20}} \right)^{99} \cdot \dfrac{80x^{19}}{(1 - 2x^{20})^2}$

(h) $f'(x) = \dfrac{-4x}{(3 - 2x^2)\ln(3 - 2x^2)}$

(i) $f'(x) = \dfrac{-2xe^{x^2}}{\sqrt{1 - e^{2x^2}}}$

(j) $f'(x) = \dfrac{x - 1}{x^2\sqrt{2x^2 - 2x + 1}}$

(k) $f'(x) = \dfrac{1}{\sqrt{1 + e^x}}$

(l) $f'(x) = \dfrac{2x + 11}{(x-2)(x+1)}$

△

Logaritamsko deriviranje

Funkciju oblika $y(x) = u(x)^{v(x)}$, gdje je $u(x) > 0$ možemo derivirati na sljedeći način:

$$\begin{aligned}
y(x) &= u(x)^{v(x)} && /\ln \\
\ln y(x) &= v(x) \ln u(x) && /' \\
\frac{y'(x)}{y(x)} &= v'(x) \ln u(x) + v(x) \frac{u'(x)}{u(x)} \\
y'(x) &= y(x) \left(v'(x) \ln u(x) + v(x) \frac{u'(x)}{u(x)} \right)
\end{aligned}$$

Primjer. Derivirajmo funkciju $f(x) = x^x,\ x > 0$.

Sjetimo se da je po definiciji $f(x) = e^{x \ln x}$. Tada je $f'(x) = e^{x \ln x}(\ln x + 1) = x^x(\ln x + 1)$.

Koristeći logaritamsko deriviranje, funkciju možemo derivirati i na sljedeći način:

$$\begin{aligned} y(x) &= x^x & /\ln \\ \ln y(x) &= x\ln x & /' \\ \frac{y'(x)}{y(x)} &= \ln x + 1 \\ \Longrightarrow\ y'(x) &= y(x)(\ln x + 1) = x^x(\ln x + 1) & . \end{aligned}$$

△

Zadatak 1.5 Derivirajte sljedeće funkcije pomoću logaritamskog deriviranja:

(a) $y(x) = (\sin x)^{\cos x}$, za $\sin x > 0$

(b) $y(x) = x^3 e^{x^2} \sin 2x$

(c) $y(x) = x^{\frac{1}{x}}$, za $x > 0$

(d) $y(x) = x^{x^x}$, za $x > 0$

(e) $y(x) = \left(1 + \frac{1}{x}\right)^x$

(f) $y(x) = \frac{(x+1)^3 \sqrt[4]{x-2}}{\sqrt[5]{(x-3)^2}}$

Rješenje.

(a)

$$\begin{aligned} y(x) &= (\sin x)^{\cos x} & /\ln \\ \ln y(x) &= \cos x \ln \sin x & /' \\ \frac{y'(x)}{y(x)} &= -\sin x \ln \sin x + \frac{\cos^2 x}{\sin x} \\ \Longrightarrow\ y'(x) &= -(\sin x)^{\cos x + 1} \ln \sin x + \cos^2 x (\sin x)^{\cos x - 1} & . \end{aligned}$$

(b)

$$\begin{aligned} |y(x)| &= |x^3 e^{x^2} \sin 2x| & /\ln \\ \ln |y(x)| &= \ln |x^3| + \ln |e^{x^2}| + \ln |\sin 2x| & /' \\ \frac{y'(x)}{y(x)} &= \frac{1}{x^3} \cdot 3x^2 + 2x + \frac{1}{\sin 2x} \cos 2x \cdot 2 \\ \Longrightarrow\ y'(x) &= x^3 e^{x^2} \sin 2x (\frac{3}{x} + 2x + 2\operatorname{tg} 2x) & . \end{aligned}$$

(c)

$$\begin{aligned} y(x) &= x^{\frac{1}{x}} && /\ln \\ \ln y(x) &= \frac{\ln x}{x} && /' \\ \frac{y'(x)}{y(x)} &= \frac{1-\ln x}{x^2} \\ \implies y'(x) &= x^{\frac{1}{x}-2}(1-\ln x) && . \end{aligned}$$

(d)

$$\begin{aligned} y(x) &= x^{x^x} && /\ln \\ \ln y(x) &= x^x \ln x && /' \\ \frac{y'(x)}{y(x)} &= x^x(\ln x + 1)\ln x + x^{x-1} \\ \implies y'(x) &= x^{x^x}(x^x(\ln x+1)\ln x + x^{x-1}) && . \end{aligned}$$

(e)

$$\begin{aligned} y(x) &= \left(1+\frac{1}{x}\right)^x && /\ln \\ \ln y(x) &= x\ln\left(1+\frac{1}{x}\right) && /' \\ \frac{y'(x)}{y(x)} &= \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1} \\ \implies y'(x) &= \left(1+\frac{1}{x}\right)^x \left(\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right) && . \end{aligned}$$

(f)

$$\begin{aligned} |y(x)| &= \left|\frac{(x+1)^3\sqrt[4]{x-2}}{\sqrt[5]{(x-3)^2}}\right| && /\ln \\ \ln y(x) &= 3\ln|x+1| + \frac{1}{4}\ln|x-2| - \frac{2}{5}\ln|x-3| && /' \\ \frac{y'(x)}{y(x)} &= \frac{3}{x+1} + \frac{1}{4(x-2)} - \frac{2}{5(x-3)} \\ \implies y'(x) &= \frac{(x+1)^3\sqrt[4]{x-2}}{\sqrt[5]{(x-3)^2}}\left(\frac{3}{x+1} + \frac{1}{4(x-2)} - \frac{2}{5(x-3)}\right) && . \end{aligned}$$

△

Zadaci za vježbu

1.6 Derivirajte sljedeće funkcije:

(a) $f(x) = \arcsin \dfrac{1}{x^2}$

(b) $f(x) = \sqrt{e^{10x}}$

(c) $f(x) = \left(\dfrac{\arcsin x}{\arccos x}\right)^5$

(d) $f(x) = \sin((\sin x)^{\sin x})$, za $\sin x > 0$

(e) $f(x) = \underbrace{x\sqrt{x\sqrt{x\sqrt{\cdots x\sqrt{x}}}}}_{n \text{ korijena}}$

(f) $f(x) = \ln \sqrt[4]{\dfrac{x^2+x+1}{x^2-x+1}} + \dfrac{1}{2\sqrt{3}}\left(\operatorname{arctg}\dfrac{2x+1}{\sqrt{3}} + \operatorname{arctg}\dfrac{2x-1}{\sqrt{3}}\right)$

(g) $f(x) = \left(x^{x^x}\right)^{x^x}$, za $x > 0$

(h) $f(x) = (\sin x - 1)^{\cos x + 1} + (\sin x + 1)^{1 - \cos x}$

(i) $f(x) = \sqrt{\ln \cos \arcsin x}$

1.7 Derivirajte sljedeće funkcije:

(a) $f(x) = \mathrm{e}^{\sqrt{\frac{1-\sin x}{1+\sin x}}}$

(b) $f(x) = \ln\left(2 + e^{-x} + \sqrt{e^x + e^{-x} + 4}\right)$

(c) $f(x) = (\arcsin x)^{\operatorname{arctg} x}$

(d) $f(x) = x \sin^2 3x \cos^3 \frac{x}{2}$

(e) $f(x) = \ln \frac{x+\cos\sqrt{\pi x}}{x-\cos\sqrt{\pi x}} + \frac{1}{3}\arcsin \ln 2x$

1.8 Za koji $a \in \mathbb{R}$ funkcija $f(x) = \frac{1+\ln x}{x - x\ln x}$ zadovoljava jednakost

$$2x^2 f'(x) - x^2 f(x)^2 = a?$$

1.9 Pokažite da funkcija $f(x) = \frac{1}{1+x+\ln x}$ zadovoljava jednakost

$$xf'(x) = f(x)(f(x)\ln x - 1).$$