1.5 L'Hôpitalovo pravilo

Teorem. (**L'Hôpitalovo pravilo**) Neka je $I \subseteq \mathbb{R}$ otvoreni interval (konačan ili beskonačan), $c \in I$ (može biti i $c = \pm\infty$ u slučaju beskonačnog intervala I) i neka su $f, g\colon I \to \mathbb{R}$ derivabilne funkcije.

1. Ako je $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$, $g'(x) \neq 0$ za sve $x \in I$ i ako postoji $\lim\limits_{x\to c} \frac{f'(x)}{g'(x)}$ u $\overline{\mathbb{R}}$, onda vrijedi
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

2. Ako je $\lim\limits_{x\to c} f(x) = \pm\infty$, $\lim\limits_{x\to c} g(x) = \pm\infty$, $g'(x) \neq 0$ za sve $x \in I$ i ako postoji $\lim\limits_{x\to c} \frac{f'(x)}{g'(x)}$ u $\overline{\mathbb{R}}$, onda vrijedi
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Zadatak 1.67 Izračunajte

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

(b) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$

(c) $\lim\limits_{x\to +\infty} xe^{-x}$

(d) $\lim\limits_{x\to +\infty} \dfrac{5x^3-4x+3}{2x^2-1}$

(e) $\lim\limits_{x\to +\infty} \dfrac{\ln(1+e^x)}{1+x}$

(f) $\lim\limits_{x\to 0} \arcsin x \cdot \operatorname{ctg} x$

(g) $\lim\limits_{x\to +\infty} \dfrac{\ln x}{\sqrt{x}}$

Rješenje.

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \left(\dfrac{0}{0}\right) \stackrel{\text{L'H}}{=} \lim\limits_{x\to 0} \dfrac{\cos x}{1} = 1$

(b) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \left(\dfrac{0}{0}\right) \stackrel{\text{L'H}}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{2x} = \left(\dfrac{0}{0}\right) \stackrel{\text{L'H}}{=} \lim\limits_{x\to 0} \dfrac{\cos x}{2} = \dfrac{1}{2}$

(c) $\lim\limits_{x\to+\infty} xe^{-x} = \lim\limits_{x\to+\infty} \frac{x}{e^x} = \left(\frac{\infty}{\infty}\right) \overset{\text{L'H}}{=} \lim\limits_{x\to+\infty} \frac{1}{e^x} = 0$

(d) $\lim\limits_{x\to+\infty} \frac{5x^3-4x+3}{2x^2-1} = \left(\frac{\infty}{\infty}\right) \overset{\text{L'H}}{=} \lim\limits_{x\to+\infty} \frac{15x^2-4}{4x} = \left(\frac{\infty}{\infty}\right) \overset{\text{L'H}}{=} \lim\limits_{x\to+\infty} \frac{30x}{4} = +\infty$

(e) $\lim\limits_{x\to+\infty} \frac{\ln(1+e^x)}{1+x} = \left(\frac{\infty}{\infty}\right) \overset{\text{L'H}}{=} \lim\limits_{x\to+\infty} \frac{e^x}{1+e^x} = \left(\frac{\infty}{\infty}\right) \overset{\text{L'H}}{=} \lim\limits_{x\to+\infty} \frac{e^x}{e^x} = 1$

(f) $\lim\limits_{x\to0} \arcsin x \cdot \operatorname{ctg} x = (0\cdot\infty) = \lim\limits_{x\to0} \frac{\arcsin x}{\operatorname{tg} x} = \left(\frac{0}{0}\right) \overset{\text{L'H}}{=} \lim\limits_{x\to0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{\cos^2 x}} = \lim\limits_{x\to0} \frac{\cos^2 x}{\sqrt{1-x^2}} = 1$

(g) $\lim\limits_{x\to+\infty} \frac{\ln x}{\sqrt{x}} = \left(\frac{\infty}{\infty}\right) \overset{\text{L'H}}{=} \lim\limits_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to+\infty} \frac{2}{\sqrt{x}} = 0$

△

Zadatak 1.68 Može li se primijeniti L'Hôpitalovo pravilo na

$$\lim_{x\to+\infty} \frac{x-\sin x}{x+\sin x}?$$

Rješenje. Ne možemo primijeniti L'Hôpitalovo pravilo, jer limes

$$\lim_{x\to+\infty} \frac{(x+\sin x)'}{(x-\sin x)'} = \lim_{x\to+\infty} \frac{1+\cos x}{1-\cos x}$$

ne postoji, jer je za $x_n = \frac{\pi}{2} + 2n\pi$ i $y_n = (2n+1)\pi$

$$\lim_{n\to+\infty} \frac{1+\cos x_n}{1-\cos x_n} = \lim_{n\to+\infty} \frac{1+0}{1-0} = 1,$$
$$\lim_{n\to+\infty} \frac{1+\cos y_n}{1-\cos y_n} = \lim_{n\to+\infty} \frac{1+(-1)}{1-(-1)} = 0.$$

Međutim,

$$\lim_{x\to+\infty} \frac{x+\sin x}{x-\sin x} = \lim_{x\to+\infty} \frac{1+\frac{\sin x}{x}}{1-\frac{\sin x}{x}} = 1.$$

△

Zadatak 1.69 Izračunajte

(a) $\lim\limits_{x\to0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

(b) $\lim\limits_{x\to0} \left(\frac{1}{\ln(x+1)} - \frac{1}{x}\right)$

(c) $\lim\limits_{x\to 0+} x^x$

(d) $\lim\limits_{x\to +\infty} x^{\frac{1}{x}}$

(e) $\lim\limits_{x\to 1}(1-x)^{\cos\frac{\pi}{2}x}$

(f) $\lim\limits_{x\to +\infty}(1+x^2)^{\frac{1}{x}}$

Rješenje.

(a) $\lim\limits_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim\limits_{x\to 0}\frac{x-\sin x}{x\sin x}=\left(\frac{0}{0}\right)\overset{\text{L'H}}{=}\lim\limits_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}=$
$=\left(\frac{0}{0}\right)\overset{\text{L'H}}{=}\lim\limits_{x\to 0}\frac{\sin x}{\cos x+\cos x+x\sin x}=0$

(b) $\lim\limits_{x\to 0}\left(\frac{1}{\ln(x+1)}-\frac{1}{x}\right)=\lim\limits_{x\to 0}\frac{x-\ln(x+1)}{x\ln(x+1)}=\left(\frac{0}{0}\right)\overset{\text{L'H}}{=}\lim\limits_{x\to 0}\frac{1-\frac{1}{x+1}}{\ln(x+1)+\frac{x}{x+1}}=$
$=\lim\limits_{x\to 0}\frac{x}{(x+1)\ln(x+1)+x}=\left(\frac{0}{0}\right)\overset{\text{L'H}}{=}\lim\limits_{x\to 0}\frac{1}{\ln(x+1)+1+1}=\frac{1}{2}$

(c) Neka je $y(x)=x^x$. Tada je $\ln y(x)=x\ln x$ pa je

$\lim\limits_{x\to 0+}\ln y(x)=\lim\limits_{x\to 0+}x\ln x=(0\cdot\infty)=\lim\limits_{x\to 0+}\frac{\ln x}{\frac{1}{x}}=\left(\frac{\infty}{\infty}\right)\overset{\text{L'H}}{=}\lim\limits_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=$
$=\lim\limits_{x\to 0+}(-x)=0.$

Zbog neprekidnosti dobijemo

$\lim\limits_{x\to 0+}y(x)=\lim\limits_{x\to 0+}e^{\ln y(x)}=e^{\lim_{x\to 0+}\ln y(x)}=e^0=1.$

(d) Neka je $y(x)=x^{\frac{1}{x}}$. Tada je $\ln y(x)=\frac{\ln x}{x}$ pa je

$\lim\limits_{x\to +\infty}\ln y(x)=\lim\limits_{x\to +\infty}\frac{\ln x}{x}=\left(\frac{\infty}{\infty}\right)\overset{\text{L'H}}{=}\lim\limits_{x\to +\infty}\frac{\frac{1}{x}}{1}=0.$

Zbog neprekidnosti dobijemo

$\lim\limits_{x\to +\infty}y(x)=\lim\limits_{x\to 1}e^{\ln y(x)}=e^{\lim_{x\to +\infty}\ln y(x)}=e^0=1.$

(e) Neka je $y(x)=(1-x)^{\cos\frac{\pi}{2}x}$. Tada je $\ln y(x)=\cos\frac{\pi}{2}x\ln(1-x)$ pa je

$\lim\limits_{x\to 1}\ln y(x)=\lim\limits_{x\to 1}\cos\frac{\pi}{2}x\ln(1-x)=(0\cdot\infty)=\lim\limits_{x\to 1}\frac{\ln(1-x)}{\frac{1}{\cos\frac{\pi}{2}x}}=\left(\frac{\infty}{\infty}\right)\overset{\text{L'H}}{=}$
$=\lim\limits_{x\to 1}\frac{\frac{-1}{1-x}}{\frac{-1}{\cos^2\frac{\pi}{2}x}\cdot\left(-\sin\frac{\pi}{2}x\cdot\frac{\pi}{2}\right)}=\frac{2}{\pi}\lim\limits_{x\to 1}\frac{\operatorname{ctg}\frac{\pi}{2}x\cos\frac{\pi}{2}x}{x-1}=\left(\frac{0}{0}\right)\overset{\text{L'H}}{=}$
$=\lim\limits_{x\to 1}\frac{-\frac{\cos\frac{\pi}{2}x}{\sin^2\frac{\pi}{2}x}\cdot\frac{\pi}{2}+\operatorname{ctg}\frac{\pi}{2}x\left(-\sin\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}}{1}=0.$

Zbog neprekidnosti dobijemo

$$\lim_{x\to 1} y(x) = \lim_{x\to 1} e^{\ln y(x)} = e^{\lim_{x\to 1}\ln y(x)} = e^0 = 1.$$

(f) Neka je $y(x) = (1+x^2)^{\frac{1}{x}}$. Tada je $\ln y(x) = \frac{\ln(1+x^2)}{x}$ pa je

$$\lim_{x\to+\infty} \ln y(x) = \lim_{x\to+\infty} \frac{\ln(1+x^2)}{x} = \left(\frac{\infty}{\infty}\right) \stackrel{\text{L'H}}{=} \lim_{x\to+\infty} \frac{\frac{2x}{1+x^2}}{1} = 0.$$

Zbog neprekidnosti dobijemo

$$\lim_{x\to+\infty} y(x) = \lim_{x\to+\infty} e^{\ln y(x)} = e^{\lim_{x\to+\infty}\ln y(x)} = e^0 = 1.$$

△

Zadatak 1.70 Izračunajte

(a) $\displaystyle\lim_{x\to 0} \frac{\arcsin x - \operatorname{arctg} x}{x^3}$

(b) $\displaystyle\lim_{x\to 1} \ln x \ln(x-1)$

(e) $\displaystyle\lim_{x\to+\infty} \frac{e^{ax}}{x^n},\ n\in\mathbb{N}, a>0$

Rješenje.

(a) $$\lim_{x\to 0} \frac{\arcsin x - \operatorname{arctg} x}{x^3} = \left(\frac{0}{0}\right) \stackrel{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}} - \frac{1}{1+x^2}}{3x^2} = \left(\frac{0}{0}\right) \stackrel{\text{L'H}}{=}$$
$$= \lim_{x\to 0} \frac{\frac{x}{(1-x^2)^{3/2}} + \frac{2x}{(1+x^2)^2}}{6x} = \frac{1}{2}$$

(b) $$\lim_{x\to 1} \ln x \ln(x-1) = (\infty\cdot 0) = \lim_{x\to 1} \frac{\ln(x-1)}{\frac{1}{\ln x}} = \left(\frac{\infty}{\infty}\right) \stackrel{\text{L'H}}{=} \lim_{x\to 1} \frac{\frac{1}{x-1}}{\frac{-1}{x\ln^2 x}} =$$
$$= \lim_{x\to 1} \frac{-x\ln^2 x}{x-1} = \left(\frac{0}{0}\right) \stackrel{\text{L'H}}{=} \lim_{x\to 1} \frac{-\ln^2 x - 2x\ln x\cdot\frac{1}{x}}{1} = 0$$

(c) $$\lim_{x\to+\infty} \frac{e^{ax}}{x^n} = \left(\frac{\infty}{\infty}\right) \stackrel{\text{L'H}}{=} \lim_{x\to+\infty} \frac{ae^{ax}}{nx^{n-1}} = \stackrel{\text{L'H}}{\cdots} = \lim_{x\to+\infty} \frac{a^n e^{ax}}{n!} = +\infty$$

△

Zadaci za vježbu

1.71 Izračunajte

(a) $\lim\limits_{x\to 0} \dfrac{\ln x}{\operatorname{ctg} x}$

(b) $\lim\limits_{x\to 2} \dfrac{x^3-2x^2-x+2}{x^3-7x+6}$

(c) $\lim\limits_{x\to 0} \dfrac{\operatorname{ch} x-1}{1-\cos x}$

(d) $\lim\limits_{x\to 0}(1-\cos x)\operatorname{ctg} x$

1.72 Izračunajte

(a) $\lim\limits_{x\to 1} \left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$

(b) $\lim\limits_{x\to 0} x^{\frac{3}{4+\ln x}}$

(c) $\lim\limits_{x\to +\infty} \dfrac{\ln x}{\sqrt[n]{x}}$, $n\in\mathbb{N}$

(d) $\lim\limits_{x\to +\infty} \dfrac{(\ln x)^n}{x}$, $n\in\mathbb{N}$

1.73 Izračunajte

(a) $\lim\limits_{x\to +\infty} \left(\cos\dfrac{1}{x}\right)^x$

(b) $\lim\limits_{x\to 0}(\sin x)^{\operatorname{tg} x}$

(c) $\lim\limits_{x\to 0} \dfrac{x+\sin 2x}{x-\sin x}$

(d) $\lim\limits_{x\to 1} \dfrac{\ln x}{\sin \pi x}$

1.74 Izračunajte

(a) $\lim\limits_{x\to +\infty} \dfrac{x+\operatorname{arctg} x}{x^3+x^2+x}$

(b) $\lim\limits_{x\to 0} \dfrac{\arcsin\sqrt{\sin x}}{\sqrt{2x-x^2}}$

(c) $\lim\limits_{x\to\pi}(\pi - x)\operatorname{tg}\dfrac{x}{2}$, $n \in \mathbb{N}$

(d) $\lim\limits_{x\to+\infty}\dfrac{\ln(1+x) - \frac{x^2}{2} + x}{e^x - \cos x}$, $n \in \mathbb{N}$

1.75 Može li se primijeniti L'Hôpitalovo pravilo na

$$\lim_{x\to 0}\frac{x^2 \sin\frac{1}{x}}{\sin x}?$$

Izračunajte gornji limes.

1.76 Izračunajte

(a) $\lim\limits_{x\to 0}\left(\dfrac{1}{x\sin x} - \dfrac{1}{x^2}\right)$

(b) $\lim\limits_{x\to+\infty}\left(\dfrac{\pi}{2} - \operatorname{arctg} x\right)^{\frac{1}{\ln x}}$

(c) $\lim\limits_{x\to+\infty}\left(\sqrt{1 + x^2\ln\dfrac{ex}{x+1}} - x\right)$

(d) $\lim\limits_{x\to+\infty}\left(x - x^2\ln\left(1 + \dfrac{1}{x}\right)\right)$