2.4 Integrali iracionalnih funkcija

Osnovna ideja pri računanju integrala iracionale funkcije je naći pogodnu supstituciju (ukoliko je to moguće) kojom bi se taj integral sveo na integral neke racionalne funkcije.

Zadatak 2.36 Izračunajte integrale:

$$\text{(a)} \int_{-1}^{0} \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}\,dx \qquad \text{(b)} \int \frac{\sqrt{x}}{\sqrt{2x^5-x^4}+\sqrt[4]{2x^{10}-x^9}}\,dx \qquad \text{(c)} \int_{2}^{3} \sqrt{\frac{x+1}{x-1}}\,dx.$$

Rješenje.

$$\text{(a)} \int_{-1}^{0} \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}\,dx = \left[\begin{array}{ll} t=\sqrt[6]{x+1} \Rightarrow x=t^6-1 & 0\mapsto 1 \\ dx=6t^5\,dt & -1\mapsto 0 \end{array}\right] = 6\int_0^1 \frac{t^5-t^8}{1+t^2}\,dt =$$

$$[\text{dijelimo polinome}: \ -t^8+t^5=(1-t-t^2+t^3+t^4-t^6)(1+t^2)+(t-1)]$$

$$6\int_0^1 (1-t-t^2+t^3+t^4-t^6)\,dt + 6\int_0^1 \frac{t\,dt}{1+t^2} - 6\int_0^1 \frac{dt}{1+t^2} = 6\Big(t-\frac{t^2}{2}-\frac{t^3}{3}+\frac{t^4}{4}+$$
$$\frac{t^5}{5}-\frac{t^7}{7}\Big)\Big|_0^1 + 3\ln(1+t^2)\Big|_0^1 - 6\,\mathrm{arctg}\,t\Big|_0^1 = \frac{199}{70}+3\ln 2-\frac{3\pi}{2}.$$

$$\text{(b)} \int \frac{\sqrt{x}}{\sqrt{2x^5-x^4}+\sqrt[4]{2x^{10}-x^9}}\,dx = \int \frac{\sqrt{x}}{x^2(\sqrt{2x-1}+\sqrt[4]{2x^2-x})} = \int \frac{dx}{x^2\left(\sqrt{2-\frac{1}{x}}+\sqrt[4]{2-\frac{1}{x}}\right)}$$

$$= \left[\begin{array}{l} t=\sqrt[4]{2-\frac{1}{x}} \Rightarrow 2-\frac{1}{x}=t^4 \\ \frac{dx}{x^2}=4t^3\,dt \end{array}\right] = 4\int \frac{t^3}{t^2+t}\,dt = 4\int \frac{t^2}{t+1}\,dt = 4\left(\int (t-1)\,dt + \int \frac{dt}{t+1}\right)$$

$$= 2t^2-4t+\ln|1+t|+C = 2\sqrt{2-\frac{1}{x}} - 4\sqrt[4]{2-\frac{1}{x}} + \ln\left(1+\sqrt[4]{2-\frac{1}{x}}\right)+C.$$

$$\text{(c)} \int_2^3 \sqrt{\frac{x+1}{x-2}}\,dx = \left[\begin{array}{ll} t=\sqrt{\frac{x+1}{x-1}} \Rightarrow x=\frac{t^2+1}{t^2-1} & 2\mapsto\sqrt{3} \\ dx=\frac{-4t}{(t^2-1)}\,dt & 3\mapsto\sqrt{2} \end{array}\right] = -4\int_{\sqrt{3}}^{\sqrt{2}} \frac{t^2}{(t^2-1)^2}\,dt$$

$$= 4\int_{\sqrt{2}}^{\sqrt{3}} \frac{t^2}{(t^2-1)^2}\,dt = \left[\begin{array}{ll} u=t & du=dt \\ dv=\frac{t}{(t^2-1)^2}\,dt & v=-\frac{1}{2(1-t^2)} \end{array}\right] = 2\Big[\Big(\frac{t}{1-t^2}\Big)\Big|_{\sqrt{2}}^{\sqrt{3}} + 2\int_{\sqrt{2}}^{\sqrt{3}} \frac{dt}{t^2-1}\Big] =$$

$$-\sqrt{3}+2\sqrt{2}+2\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\Big|_{\sqrt{2}}^{\sqrt{3}} = -\sqrt{3}+2\sqrt{2}+2\ln\frac{\sqrt{3}-1}{2}-2\ln(\sqrt{2}-1).$$

Napomena. Prethodni zadatak možemo lako poopćiti. Neka je R racionalna funkcija i neka su $m_1,\ldots,m_k$ cijeli brojevi te $n_1,\ldots,n_k$ prirodni brojevi. Stavimo

$$N := \mathrm{NZV}\{n_1,\ldots,n_k\}.$$

Ako su $a, b, c, d \in \mathbb{R}$ takvi da je $ad - bc \neq 0$, tada integral oblika

$$\int R\left(x, \sqrt[n_1]{\left(\frac{ax+b}{cx+d}\right)^{m_1}}, \ldots, \sqrt[n_k]{\left(\frac{ax+b}{cx+d}\right)^{m_k}}\right) dx$$

možemo svesti na integral racionalne funkcije koristeći supstituciju

$$t := \sqrt[N]{\frac{ax+b}{cx+d}}.$$

Zadatak 2.37 Izračunajte integrale:

$$\text{(a)} \int_{-3}^{3} \sqrt{9-x^2}\, dx \qquad \text{(b)} \int \sqrt{1+x^2}\, dx \qquad \text{(c)} \int_{-4}^{-2} \sqrt{x^2-4}\, dx.$$

Rješenje.

(a) $$\int_{-3}^{3} \sqrt{9-x^2}\, dx = [\text{podintegralna funkcija je parna}] = 2\int_0^3 \sqrt{9-x^2}\, dx =$$

$$\left[\begin{array}{ll} x = 3\sin t \Rightarrow t = \arcsin\frac{x}{3} & 0 \to 0 \\ dx = 3\cos t\, dt & 3 \to \frac{\pi}{2} \end{array}\right] = 3\int_0^{\frac{\pi}{2}} \sqrt{9(1-\sin^2 t)}\cos t\, dt = 9\int_0^{\frac{\pi}{2}} \cos^2 t\, dt$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2t)\, dt = 9\left(t + \frac{\sin 2t}{2}\right)\Big|_0^{\frac{\pi}{2}} = \frac{9\pi}{2}.$$

(b) $$\int \sqrt{1+x^2}\, dx = \left[\begin{array}{l} x = \operatorname{sh} t \Rightarrow t = \operatorname{Arsh} x \\ dx = \operatorname{ch} t\, dt \end{array}\right] = \int \sqrt{1+\operatorname{sh}^2 t}\operatorname{ch} t\, dt = \int \operatorname{ch}^2 t\, dt =$$
$$\frac{1}{2}\int (\operatorname{ch} 2t + 1)\, dt = \frac{1}{2}\left(\frac{\operatorname{sh} 2t}{2} + t\right) + C = \frac{1}{2}\left(x\sqrt{1+x^2} + \operatorname{Arsh} x\right) + C.$$

(c) $$\int_{-4}^{-2} \sqrt{x^2-4}\, dx = [\text{podintegralna funkcija je parna}] = \int_2^4 \sqrt{x^2-4}\, dx =$$

$$\left[\begin{array}{ll} x = 2\operatorname{ch} t \Rightarrow t = \operatorname{Arch}\frac{x}{2} & 2 \to 0 \\ dx = 2\operatorname{sh} t\, dt & 4 \to \operatorname{Arch} 2 \end{array}\right] = 2\int_0^{\operatorname{Arch} 2} \sqrt{4(\operatorname{ch}^2 t - 1)}\operatorname{sh} t\, dt = 4\int_0^{\operatorname{Arch} 2} \operatorname{sh}^2 t\, dt =$$

$$2\int_0^{\operatorname{Arch} 2} (\operatorname{ch} 2t - 1)\, dt = 2\left(\frac{\operatorname{sh} 2t}{2} - t\right)\Big|_0^{\operatorname{Arch} 2} = 4\sqrt{3} - 2\operatorname{Arch} 2.$$

Napomena. Prethodni zadatak možemo lako generalizirati. Neka je R racionalna funkcija.

(1) Integral tipa

$$\int R(x, \sqrt{k^2 - x^2})\, dx, \quad k > 0$$

supstitucijom

$$x = k \sin t \quad \text{odnosno} \quad x = k \operatorname{th} t$$

svodimo na integral trigonometrijskih odnosno hiperbolnih funkcija.

(2) integral tipa

$$\int R(x, \sqrt{x^2 + k^2})\, dx, \quad k > 0$$

supstitucijom

$$x = k \operatorname{tg} t \quad \text{odnosno} \quad x = k \operatorname{sh} t$$

svodimo na integral trigonometrijskih odnosno hiperbolnih funkcija.

(3) integral tipa

$$\int R(x, \sqrt{k^2 - k^2})\, dx, \quad k > 0$$

supstitucijom

$$x = \frac{k}{\cos t} \quad \text{odnosno} \quad x = k \operatorname{ch} t,$$

svodimo na integral trigonometrijskih odnosno hiperbolnih funkcija.

Općenito, integral oblika

$$\int R(x, \sqrt{ax^2 + bx + c})\, dx,$$

svodimo na jedan od tri prethodno navedena tipa integrala koristeći supstituciju

$$t = x + \frac{b}{2a}.$$

Zadatak 2.38 Izračunajte integrale:

$$\text{(a)} \int_0^1 x^2 \sqrt{x - x^2}\, dx \qquad \text{(b)} \int \frac{5x + 4}{\sqrt{x^2 + 2x + 5}}\, dx.$$

Rješenje.

(a) $\int\limits_0^1 x^2\sqrt{x-x^2}\,dx = \int\limits_0^1 x^2\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}\,dx = \left[\begin{array}{ll} t = x-\frac{1}{2} \Rightarrow x = t+\frac{1}{2} & 0 \to -\frac{1}{2} \\ dt = dx & 1 \to \frac{1}{2} \end{array}\right] =$
$\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left(t+\frac{1}{2}\right)^2 \sqrt{\frac{1}{4}-t^2}\,dt = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left(t^2+\frac{1}{4}\right)\sqrt{\frac{1}{4}-t^2}\,dt = 2\int\limits_0^{\frac{1}{2}} \left(t^2+\frac{1}{4}\right)\sqrt{\frac{1}{4}-t^2}\,dt$
$= \left[\begin{array}{ll} t = \frac{1}{2}\sin s \Rightarrow s = \arcsin 2t & 0 \to 0 \\ dt = \frac{1}{2}\cos s\,ds & \frac{1}{2} \to \frac{\pi}{2} \end{array}\right] = \frac{1}{8}\int\limits_0^{\frac{\pi}{2}} (\sin^2 s+1)\cos^2 s\,ds$
$= \frac{1}{8}\left(\int\limits_0^{\frac{\pi}{2}} \frac{\sin^2 2s}{4}\,ds + \int\limits_0^{\frac{\pi}{2}} \frac{1+\cos 2s}{2}\,ds\right) = \frac{1}{8}\left(\int\limits_0^{\frac{\pi}{2}} \frac{1-\sin 4s}{8}\,ds + \int\limits_0^{\frac{\pi}{2}} \frac{1+\cos 2s}{2}\,ds\right) =$
$\cdots = \frac{5\pi}{128}.$

(b) $\int \frac{5x+4}{\sqrt{x^2+2x+5}}\,dx = \int \frac{5x+4}{\sqrt{(x+1)^2+4}}\,dx = \left[\begin{array}{l} t = x+1 \Rightarrow x = t-1 \\ dx = dt \end{array}\right] = \int \frac{5t-1}{\sqrt{t^2+4}}\,dt =$
$5\int \frac{t\,dt}{\sqrt{t^2+4}} - \int \frac{dt}{\sqrt{t^2+4}} = 5\int (\sqrt{t^2+4})'\,dt - \int \frac{dt}{\sqrt{t^2+4}} = 5\sqrt{t^2+4} - \frac{1}{2}\operatorname{Arsh}\frac{t}{2} +$
$C = 5\sqrt{x^2+2x+5} - \frac{1}{2}\operatorname{Arsh}\frac{x+1}{2} + C.$

Zadaci za vježbu

2.39 Izračunajte integrale:

$$\text{(a)} \int x\sqrt[3]{4+3x}\,dx \qquad \text{(b)} \int \frac{\sqrt{x}dx}{1+\sqrt{x}} \qquad \text{(c)} \int \frac{\sqrt{x}+1}{\sqrt{x}-1}\,dx$$

2.40 Izračunajte integrale:

$$\text{(a)} \int \sqrt{\frac{x}{2-x}}\,dx \qquad \text{(b)} \int \frac{dx}{x-\sqrt{x^2-1}} \qquad \text{(c)} \int \frac{1}{1-2x}\sqrt{\frac{1+2x}{1-2x}}\,dx$$

2.41 Izračunajte integrale:

$$\text{(a)} \int \frac{dx}{x^6\sqrt{x^2-1}} \qquad \text{(b)} \int \sqrt{2-x-x^2}\,dx \qquad \text{(c)} \int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}\,dx$$

2.42 Izračunajte integrale:

$$\text{(a)} \int_1^{16} \operatorname{arctg}\sqrt{\sqrt{x}-1}\,dx \qquad \text{(b)} \int \frac{\sqrt[14]{x^{28}(x-1)^9(x+1)^5}+14}{\sqrt[7]{(x-1)^9(x+1)^5}+14x^2}\,dx$$

$$\text{(c)} \int \ln\left(\sqrt{1-x}+\sqrt{1+x}\right)dx$$