2.5 Integrali trigonometrijskih i hiperbolnih funkcija

Zadatak 2.43 Izračunajte integrale:

(a) $\int \sin^2 x \cos^3 x \, dx$ (b) $\int_0^{\frac{\pi}{8}} \sin^4 x \cos^2 x \, dx$ (c) $\int \operatorname{sh}^3 x \, dx$ (d) $\int \cos^6 x \, dx$.

Rješenje.

(a) $\int \sin^2 x \cos^3 x \, dx = \begin{bmatrix} t = \sin x \\ dt = \cos x \, dx \end{bmatrix} = \int t^2(1-t^2)\, dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{x} - \frac{\sin^5 x}{5} + C$

(b) $\int_0^{\frac{\pi}{8}} \sin^4 x \cos^2 x \, dx = \int_0^{\frac{\pi}{8}} \sin^2 x (\sin x \cos x)^2 \, dx = \int_0^{\frac{\pi}{8}} \frac{1-\cos 2x}{2} \cdot \frac{\sin^2 2x}{4} \, dx = \frac{1}{8} \int_0^{\frac{\pi}{8}} \sin^2 2x \, dx - \frac{1}{8} \int_0^{\frac{\pi}{8}} \sin^2 2x \cos 2x \, dx = \ldots = \frac{3\pi + 2\sqrt{2} - 2}{384}$

(c) $\int \operatorname{sh}^3 x \, dx = \begin{bmatrix} t = \operatorname{ch} x \\ dt = \operatorname{sh} x \, dx \end{bmatrix} = \int (t^2 - 1)\, dt = \frac{t^3}{3} - t + C = \frac{\operatorname{ch}^3 x}{3} - \operatorname{ch} x + C$

(d) Označimo $I_n = \int \cos^n x \, dx$. Tada je

$I_n = \int \cos^n x \, dx = \begin{bmatrix} u = \cos x^{n-1} & du = -(n-1)\cos^{n-2} \sin x \, dx \\ dv = \cos x \, dx & v = \sin x \end{bmatrix} = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$ pa vrijedi rekurzivna relacija: $I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n} \cos^{n-1} x \sin x$.

Sada je $I_0 = \int dx = x + C$

$I_2 = \frac{1}{2} I_0 + \frac{\cos x \sin x}{2} = \frac{x}{2} + \frac{\cos x \sin x}{2} + C$

$I_4 = \frac{3}{4} I_2 + \frac{\cos^3 x \sin x}{4} = \frac{3x}{8} + \frac{3\cos x \sin x}{8} + \frac{\cos^3 x \sin x}{4} + C$

$I_6 = \frac{5}{6} I_4 + \frac{\cos^5 x \sin x}{6} = \frac{5x}{16} + \frac{5\cos x \sin x}{48} + \frac{5\cos^3 x \sin x}{24} + \frac{\cos^5 x \sin x}{6} + C$

△

Napomena. Neka je R racionalna funkcija.

(a) Za računanje integrala oblika $\int R(\sin x, \cos x)\, dx$ se koristi **univerzalna supstitucija**:

$$t = \operatorname{tg}\frac{x}{2} \qquad \sin x = \frac{2t}{1+t^2}$$
$$dx = \frac{2\,dt}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$$
$$x = 2\operatorname{arctg} t$$

Provjerimo supstituciju:
$dx = \dfrac{2dt}{1+t^2}$
$\sin x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2} = \dfrac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \dfrac{2t}{1+t^2}$
$\cos x = 2\cos^2\dfrac{x}{2} - 1 = \dfrac{2}{1+\operatorname{tg}^2\frac{x}{2}} - 1 = \dfrac{2}{1+t^2} - 1 = \dfrac{1-t^2}{1+t^2}$

(b) Za računanje integrala oblika $\int R(\operatorname{sh} x, \operatorname{ch} x)\, dx$ se koristi **univerzalna supstitucija**:

$$t = \operatorname{th}\frac{x}{2} \qquad \operatorname{sh} x = \frac{2t}{1-t^2}$$
$$dx = \frac{2\,dt}{1-t^2} \qquad \operatorname{ch} x = \frac{1+t^2}{1-t^2}$$
$$x = 2\operatorname{Arth} t$$

Provjerimo supstituciju:
$dx = \dfrac{2dt}{1-t^2}$
$\operatorname{sh} x = 2\operatorname{sh}\dfrac{x}{2}\operatorname{ch}\dfrac{x}{2} = \dfrac{2\operatorname{th}\frac{x}{2}}{1-\operatorname{th}^2\frac{x}{2}} = \dfrac{2t}{1-t^2}$
$\operatorname{ch} x = 2\operatorname{ch}^2\dfrac{x}{2} + 1 = \dfrac{2}{1-\operatorname{th}^2\frac{x}{2}} + 1 = \dfrac{2}{1-t^2} + 1 = \dfrac{1+t^2}{1-t^2}$

Zadatak 2.44 Izračunajte integral koristeći univerzalne supstitucije:

$$\text{(a)} \int \frac{dx}{(2+\cos x)\sin x} \qquad \text{(b)} \int \frac{dx}{1+2\operatorname{sh} x+3\operatorname{ch} x}.$$

Rješenje.

(a) $\int \frac{dx}{(2+\cos x)\sin x} = [\text{univerzalna supstitucija } \ t = \operatorname{tg}\frac{x}{2}] = \int \frac{1}{(2+\frac{1-t^2}{1+t^2})\frac{2t}{1+t^2}}\,\frac{2dt}{1+t^2} =$
$\int \frac{1+t^2}{(t^2+3)t}\,dt = [\text{rastav na parcijalne razlomke }] = \frac{1}{3}\int \frac{dt}{t} + \frac{2}{3}\int \frac{t\,dt}{t^2+3} = \frac{1}{3}\ln|t| +$
$\frac{1}{3}\ln(t^2+3) + C = \frac{1}{3}\ln|t(t^2+3)| + C = \frac{1}{3}\ln|\operatorname{tg}^3\frac{x}{2} + 3\operatorname{tg}\frac{x}{2}| + C$

(b) $\int \frac{dx}{1+2\operatorname{sh}x+3\operatorname{ch}x} = [\text{univerzalna supstitucija } \ t = \operatorname{th}\frac{x}{2}] = \int \frac{1}{(1+2\frac{2t}{1-t^2}+3\frac{1+t^2}{1-t^2})}\,\frac{2dt}{1-t^2} =$
$\int \frac{2dt}{2t^2+4t+4} = \int \frac{dt}{(t+1)^2+1} = \operatorname{arctg} t + 1 + C = \operatorname{arctg}\left(\operatorname{th}\frac{x}{2}+1\right) + C$

△

Zadaci za vježbu

2.45 Izračunajte integrale:

$$\text{(a)} \int \sin^4 x \cos^5 x \, dx \qquad \text{(b)} \int \operatorname{tg}^5 x \, dx \qquad \text{(c)} \int \frac{\cos x \, dx}{\sqrt{\sin 2x}}$$

2.46 Izračunajte integrale:

$$\text{(a)} \int \frac{dx}{\sin x - \sin a} \qquad \text{(b)} \int \frac{dx}{\sin(x+a)\sin(x+b)} \qquad \text{(c)} \int \frac{\sin x \, dx}{\sqrt{2}\sin x + \cos x}$$

2.47 Izračunajte integrale:

$$\text{(a)} \int_{-\pi/4}^{0} \frac{dx}{\cos x + 2\sin x + 3} \qquad \text{(b)} \int_{\pi/4}^{\pi/3} \frac{dx}{\sin^2 x \cos x} \qquad \text{(c)} \int_{0}^{\pi/4} \frac{dx}{\cos^4 x} \, dx$$

2.48 Izračunajte integrale:

$$\text{(a)} \int_{0}^{2\pi} \frac{dx}{5 - 4\cos x} \qquad \text{(b)} \int \frac{dx}{2\sin x - \cos x + 5} \qquad \text{(c)} \int_{0}^{2\pi} \frac{dx}{\sin^4 x + \cos^4 x} \, dx$$