Integrali

1. Izračunajte sljedeće neodređene integrale:

(1) $\int \sin^6 x \cos x \, dx$

(2) $\int \frac{dx}{e^x + e^{-x}}$

(3) $\int \frac{\operatorname{arctg} x}{1+x^2} \, dx$

(4) $\int \frac{dx}{(\arcsin x)^2 \sqrt{1-x^2}}$

(5) $\int \sqrt{\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}} \, dx$

(6) $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} \, dx$

(7) $\int \frac{dx}{x \ln x \ln(\ln x)}$

(8) $\int \frac{6^x}{9^x - 4^x} \, dx$

(9) $\int \frac{dx}{\sqrt{(x+a)(x+b)}} \quad (a \neq b)$

(10) $\int \ln x \, dx$

(11) $\int \frac{\ln x}{x} \, dx$

(12) $\int x^n \ln x \, dx \quad (n \in \mathbb{N})$

(13) $\int \left(\frac{\ln x}{x}\right)^2 dx$

(14) $\int x \operatorname{arctg} x \, dx$

(15) $\int \ln\left(x+\sqrt{1+x^2}\right) \, dx$

(16) $\int x \, \ln\left(\frac{1+x}{1-x}\right) dx$

(17) $\int x^5 e^{x^3} \, dx$

(18) $\int x \left(\operatorname{arctg} x\right)^2 \, dx$

(19) $\int \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} \, dx$

(20) $\int \frac{\arccos x}{(1-x^2)^{3/2}} \, dx$

(21) $\int \sqrt{x} \operatorname{arctg} \sqrt{x} \, dx$

(22) $\int e^{ax} \cos bx \, dx \quad (a, b \neq 0)$

(23) $\int e^{ax} \sin bx \, dx \quad (a, b \neq 0)$

(24) $\int \frac{x \, e^{\operatorname{arctg} x}}{(1+x^2)^{3/2}} \, dx$

(25) $\int \cos\left(\ln x\right) \, dx$

(26) $\int \sin\left(\ln x\right) \, dx$

2. Izračunajte sljedeće određene integrale:

(27) $\int_0^4 \frac{dx}{1+\sqrt{x}}$

(28) $\int_0^{\ln 2} \sqrt{e^x - 1} \, dx$

(29) $\int_0^{\frac{3}{4}} \frac{dx}{(x+1)\sqrt{x^2+1}}$

(30) $\int_0^{100\pi} \sqrt{1-\cos 2x} \, dx$

(31) $\int_{-1}^{1} \frac{dx}{x^2 - 2x\cos\alpha + 1} \quad (0 < \alpha < \pi)$

(32) $\int_{\frac{1}{e}}^{e} \left|\ln x\right| \, dx$

(33) $\int_1^e \left(x \ln x\right)^2 \, dx$

(34) $\int_0^{\sqrt{3}} x \operatorname{arctg} x \, dx$

3. Izračunajte sljedeće neodređene integrale:

(35) $\displaystyle\int \frac{dx}{3x^2-2x-1}$

(36) $\displaystyle\int \frac{x^3\,dx}{x^4-x^2-2}$

(37) $\displaystyle\int \frac{x\,dx}{x^4-2x^2-1}$

(38) $\displaystyle\int \frac{x\,dx}{(x-1)^2(x+1)^3}$

(39) $\displaystyle\int \frac{dx}{x^4+2x^3+3x^2+2x+1}$

(40) $\displaystyle\int \frac{dx}{(x+1)(x^2+1)(x^3+1)}$

(41) $\displaystyle\int \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}\,dx$

(42) $\displaystyle\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$

(43) $\displaystyle\int \frac{e^{2x}(6-e^{2x})}{e^{4x}+3e^{2x}-4}\,dx$

(44) $\displaystyle\int \frac{dx}{(2+\cos x)\sin x}$

(45) $\displaystyle\int \frac{dx}{2\sin x-\cos x+5}$

4. Izračunajte sljedeće neprave integrale:

(46) $\displaystyle\int_1^e \frac{dx}{x\sqrt{\ln x}}$

(47) $\displaystyle\int_0^{\ln 2} \frac{e^x dx}{\sqrt[3]{e^x-1}}$

(48) $\displaystyle\int_0^{\frac{\pi}{2}} \operatorname{tg} x\,\sqrt[n]{\cos x}\,dx \qquad (n\in\mathbb{N})$

(49) $\displaystyle\int_0^2 \frac{dx}{\sqrt{4-x^2}}$

(50) $\displaystyle\int_0^{+\infty} \frac{dx}{(1+x^2)^2}$

(51) $\displaystyle\int_1^{+\infty} \frac{dx}{x(x+1)(x+2)}$

5. Ispitajte konvergiraju li sljedeći nepravi integrali:

(52) $\displaystyle\int_0^{+\infty} \frac{\cos(x^2)}{x^2+1}\,dx$

(53) $\displaystyle\int_0^1 \frac{dx}{3\sqrt{x}+2\sqrt[3]{x}}$

(54) $\displaystyle\int_0^{+\infty} \frac{dx}{\operatorname{sh} x}$

(55) $\displaystyle\int_0^{+\infty} \frac{e^{-x}}{\sqrt[3]{x}}\,dx$

(56) $\displaystyle\int_1^{+\infty} \frac{dx}{x(x+1)...(x+9)}$

(57) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^7}}$

6. Za svaki $n\in\mathbb{N}_0$ izračunajte:

(58) $I_n=\displaystyle\int_{-1}^1 (1-x^2)^n\,dx$

(59) $I_n=\displaystyle\int_0^{+\infty} x^{2n+1}e^{-x^2}\,dx$

Rješenja:

(1) $\frac{1}{7}\sin^7 x + C$

(2) $\operatorname{arctg} e^x + C$

(3) $\frac{1}{2}(\operatorname{arctg} x)^2 + C$

(4) $-\frac{1}{\arcsin x} + C$

(5) $\frac{2}{3}\left(\ln\left(x+\sqrt{1+x^2}\right)\right)^{\frac{3}{2}} + C$

(6) $\frac{3}{2}\left(\sin x - \cos x\right)^{\frac{2}{3}} + C$

(7) $\ln|\ln(\ln x)| + C$

(8) $\frac{\ln\left(\frac{3^x-2^x}{3^x+2^x}\right)}{2\ln\frac{3}{2}} + C$

(9) $\ln\left(x+\frac{a+b}{2}+\sqrt{(x+a)(x+b)}\right) + C$

(10) $x\ln x - x + C$

(11) $\frac{1}{2}(\ln x)^2 + C$

(12) $\frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$

(13) $-\frac{(\ln x)^2+2\ln x+2}{x} + C$

(14) $\frac{x^2+1}{2}\operatorname{arctg} x - \frac{x}{2} + C$

(15) $x\ln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2} + C$

(16) $\frac{x^2-1}{2}\ln\left(\frac{1+x}{1-x}\right) + x + C$

(17) $\frac{1}{3}e^{x^3}\left(x^3-1\right) + C$

(18) $\frac{x^2+1}{2}(\operatorname{arctg} x)^2 - x\operatorname{arctg} x + \frac{1}{2}\ln\left(x^2+1\right) + C$

(19) $\sqrt{1+x^2}\ln\left(x+\sqrt{1+x^2}\right) - x + C$

(20) $\frac{x\arccos x}{\sqrt{1-x^2}} - \frac{1}{2}\ln\left(1-x^2\right) + C$

(21) $\frac{2}{3}x^{\frac{3}{2}}\operatorname{arctg}\sqrt{x} - \frac{x}{3} + \frac{1}{3}\ln(x+1) + C$

(22) $\frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C$

(23) $\frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$

(24) $\frac{e^{\operatorname{arctg} x}(x-1)}{2\sqrt{1+x^2}} + C$

(25) $\frac{x}{2}\left(\cos\left(\ln x\right) + \sin\left(\ln x\right)\right) + C$

(26) $\frac{x}{2}\left(\sin\left(\ln x\right) - \cos\left(\ln x\right)\right) + C$

(27) $4 - 2\ln 3$

(28) $2 - \frac{\pi}{2}$

(29) $\frac{\ln\left(\frac{9+4\sqrt{2}}{7}\right)}{\sqrt{2}}$

(30) $200\sqrt{2}$

(31) $\frac{\pi}{2\sin\alpha}$

(32) $2 - \frac{2}{e}$

(33) $\frac{5e^3-2}{27}$

(34) $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

(35) $\frac{1}{4}\ln\left|\frac{x-1}{x+\frac{1}{3}}\right| + C$

(36) $\frac{1}{6}\ln(x^2+1) + \frac{1}{3}\ln|x^2-2| + C$

(37) $\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-1-\sqrt{2}}{x^2-1+\sqrt{2}}\right| + C$

(38) $\frac{1}{16}\ln\left|\frac{x+1}{x-1}\right| - \frac{x^2+x+2}{8(x-1)(x+1)^2} + C$

(39) $\frac{2x+1}{3(x^2+x+1)} + \frac{4}{3\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C$

(40) $\frac{1}{3}\ln|x+1| - \frac{1}{6(x+1)} + \frac{1}{2}\operatorname{arctg} x$
$-\frac{1}{6}\ln(x^2-x+1) - \frac{1}{3\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} + C$

(41) $-\frac{6}{7}(x+1)^{\frac{7}{6}} + \frac{6}{5}(x+1)^{\frac{5}{6}} + \frac{3}{2}(x+1)^{\frac{2}{3}}$
$-2(x+1)^{\frac{1}{2}} - 3(x+1)^{\frac{1}{3}} + 6(x+1)^{\frac{1}{6}}$
$-6\operatorname{arctg}(x+1)^{\frac{1}{6}} + 3\ln((x+1)^{\frac{1}{3}}+1) + C$

(42) $-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}} + C$

(43) $\frac{1}{2}\ln|e^{2x}-1| - \ln(e^{2x}+4) + C$

(44) $\frac{1}{6}\ln(1-\cos x) - \frac{1}{2}\ln(1+\cos x)$
$+\frac{1}{3}\ln(2+\cos x) + C$

(45) $\frac{1}{\sqrt{5}}\operatorname{arctg}\frac{3\operatorname{tg}\frac{x}{2}+1}{\sqrt{5}} + C$

(46) 2

(47) $\frac{3}{2}$

(48) n

(49) $\frac{\pi}{2}$

(50) $\frac{\pi}{4}$

(51) $\frac{1}{2}\ln\frac{4}{3}$

(52) DA

(53) DA

(54) NE

(55) DA

(56) DA

(57) DA

(58) $\frac{2\cdot(2n)!!}{(2n+1)!!}$, tj. $\frac{2^{2n+1}\cdot(n!)^2}{(2n+1)!}$

(59) $\frac{n!}{2}$