## V.2.1.2.2.1. SPEKTRALNA METODA

- uzimamo za fje razvoja trigonometrijske fje i koristimo Fourierov red ⇒ tada govorimo o spektralnoj metodi

$$\hat{\phi}(x,t) = \frac{\phi_0^c(t)}{2} + \sum_{m=1}^{M} \left[ \phi_m^c(t) \cos m\lambda + \phi_m^s(t) \sin m\lambda \right] \quad ; \quad \lambda = \frac{2\pi x}{L}$$

- gornji indeksi : c ... koeficijenti uz cos
  s ... koeficijenti uz sin
- zbog jednostavnosti se prelazi na exp. izraze, ali koristi se samo realni dio

$$\cos m\lambda = \frac{e^{im\lambda} + e^{-im\lambda}}{2} \quad ; \quad \sin m\lambda = \frac{e^{im\lambda} - e^{-im\lambda}}{2i}$$

$$\Rightarrow \boxed{\hat{\phi}(x,t) = \sum_{m=-M}^{M} \phi_m(t) e^{im\lambda} \quad ; \quad \phi_0^s(t) = 0}$$

- vrijedi :

$$\left.\begin{array}{ll} \phi_m(t) = \frac{1}{2}\left[\phi_m^s(t) - i\phi_m^c(t)\right] & \text{za } m>0 \\ \phi_m(t) = \frac{1}{2}\left[\phi_m^s(t) + i\phi_m^c(t)\right] & \text{za } m<0 \end{array}\right\rangle \phi_{-m}(t) = \phi_m^*(t)$$

- koeficijenti u F. razvoju : $\phi_m(t) = \frac{1}{2\pi}\int_0^{2\pi} \hat{\phi}(x,t) e^{-im\lambda} d\lambda$

### ① PRIMJENA NA LINEARNU ADV. JDŽBU

- prevedimo tu jdžbu na kutne varijable :

$$\boxed{\frac{\partial w(\lambda,t)}{\partial t} + \mu \frac{\partial w(\lambda,t)}{\partial \lambda} = 0} \quad ; \quad w = \frac{2\pi}{L}\hat{\phi}(x,t) \quad ; \quad \lambda = \frac{2\pi}{L}x \quad ; \quad \mu = \frac{2\pi}{L}c \quad ; \quad \underbrace{L = 2\pi a}_{\text{opseg paralele}}$$

$$w(\lambda,t) = \sum_{m=-M}^{M} w_m(t) e^{im\lambda} \Rightarrow \text{~~pp~~ da se } w(\lambda,t) \text{ može razviti u F. red}$$

- ako to uvrstimo u jdžbu :

$$\sum_{m=-M}^{M} \frac{\partial w_m(t)}{\partial t} e^{im\lambda} + \mu \sum_{m=-M}^{M} im\, w_m(t) e^{im\lambda} = 0$$

diff. jdžba

$$\sum_{m=-M}^{M} \frac{\partial w_m(t)}{\partial t} \cancel{e^{im\lambda}} + \sum_{m=-M}^{M} im\mu\, w_m(t) \cancel{e^{im\lambda}} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial w_m(t)}{\partial t} + im\mu\, w_m(t) = 0}$$

- riješimo tu jdžbu : $\frac{\partial w_m(t)}{w_m(t)} = -im\mu\, \partial t \Big/ \int_{w_m(0),\,0}^{w_m(t),\,t}$

ovdje dobijemo koeficijente

$$\Rightarrow \int_{w_m(0)}^{w_m(t)} \frac{\partial w_m(t)}{w_m(t)} = -im\mu \int_0^t \partial t \Rightarrow \ln\frac{w_m(t)}{w_m(0)} = -im\mu t \Rightarrow \boxed{w_m(t) = w_m(0)\, e^{-im\mu t}}$$

$$\Rightarrow \boxed{w(x,t) = \sum_{m=-M}^{M} w_m(0)\, e^{im(\lambda - \mu t)}}$$

⇒ praktički dobili analitičko rješenje (jer je t kontinuirano)
- naravno, analitičko bi bilo za $M \to \infty$

② PRIMJENA NA NELINEARNU ADV. JDŽBU

- uvodimo nelinearan član $F = -\omega(\lambda,t)\frac{\partial \omega(\lambda,t)}{\partial\lambda}$
- problem izgleda: $\frac{\partial \omega(\lambda,t)}{\partial t} = -\omega(\lambda,t)\frac{\partial \omega(\lambda,t)}{\partial \lambda} = F(\lambda,t)$
- sada mogu razviti obje strane:

$$\omega(\lambda,t) = \sum_{m=-M}^{M} \omega_m(t)\, e^{im\lambda} \quad ; \quad F(\lambda,t) = \sum_{m=-M}^{M} F_m(t)\, e^{im\lambda}$$

$\partial \to d$ jer je $\omega_m = \omega_m(t)$ ovisi samo o $t$!

- uvrstim to u diff. jdžbu:

$$\sum_{m=-M}^{M} \frac{\partial \omega_m(t)}{\partial t} e^{im\lambda} = \sum_{m=-M}^{M} F_m(t)\, e^{im\lambda} \Rightarrow \boxed{F_m(t) = \frac{\partial \omega_m(t)}{\partial t} = \frac{d\omega_m}{dt} \; ; \; m\in[-M,M]}$$

- ovo se može riješiti na 2 načina:

(a) METODA KOEFICIJENATA INTERAKCIJE

$$\frac{d\omega_m(t)}{dt} = -\sum_k \sum_l i\omega_k \omega_l I_{klm} \quad ; \quad \boxed{I_{klm} = \int e^{ik\lambda} e^{il\lambda} e^{-im\lambda}\, d\lambda}$$

→ koef. interakcije konstantni u $t$

→ danas se malo koristi

(b) METODA BRZE FOURIEROVE TRANSFORMAC. (FFT)

- znamo: $\omega(\lambda_j,t) = \sum_m \omega_m(t)\, e^{im\lambda_j}$ ; $D = \frac{\partial \omega(\lambda_j,t)}{\partial \lambda_j} = D(\lambda_j,t)$

$$\Rightarrow D = \sum_m im\,\omega_m(t)\, e^{im\lambda_j}$$

- iz def. nelinearnog člana: $F(\lambda_j) = -\omega(\lambda_j)\, D(\lambda_j)$
- povratak u spektr. prostor: $F_m = \frac{1}{2\pi}\sum_j F(\lambda_j)\, e^{-im\lambda_j}$

## V.2.1.2.2.2. PSEUDOSPEKTRALNA METODA

- ~~M~~. razvoj samo prostorno (zato „pseudo"):

$$\omega(\lambda_j,t_0) = \sum_{m=-M}^{M} \omega_m(t_0)\, e^{im\lambda_j} \quad ; \quad t_0 \ldots \text{neki poč. trenutak}$$

$$\omega_m(t_0) = \sum_{j=1}^{J} \omega(\lambda_j,t_0)\, e^{-im\lambda_j}$$

$$D(\lambda_j,t_0) = \sum_{m=-M}^{M} im\,\omega_m(t_0)\, e^{im\lambda_j}.$$

- naše rješenje nelinearne adv. jdžbe je: $\left(\frac{\partial \omega}{\partial t}\right)_j^{t_0} = -\omega(\lambda_j,t_0)\, D(\lambda_j,t_0)$

$$\left(\frac{\partial \omega}{\partial t}\right)_j^{t_0} = \frac{\omega_j(t_0+\Delta t) - \omega_j(t_0)}{\Delta t}$$

- ova metoda ima dosta nedostataka

## V.2.1.2.2.3. SFERNA GEOMETRIJA

- povoljno za praksu; u tom slučaju razmatramo met. elemente
- varijable su valnog oblika, ali ne trigonometrijske, već su reprezentirane kuglinim funkcijama

$$\Rightarrow \boxed{\phi(\lambda,\mu,t)=\sum_{m=-M}^{M}\sum_{n=|m|}^{N(m)}\phi_n^m(t)\,Y_n^m(\lambda,\mu)}$$

gdje je: $\boxed{Y_n^m(\lambda,\mu)=P_n^m(\mu)e^{im\lambda}}$ ... kuglina fja n-tog reda i m-tog stupnja

$P_n^m(\lambda,\mu)$... Legendreov polinom n-tog reda i m-tog stupnja

$\lambda$... geografska duljina ; $\mu=\sin\varphi$... geogr. širina

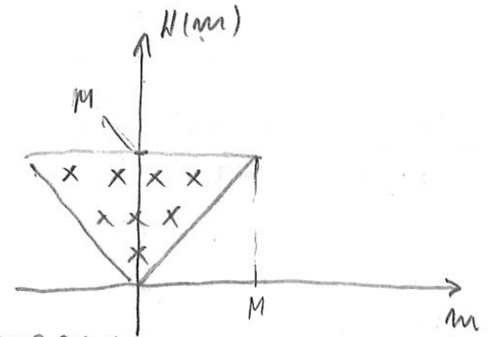
- možemo odabrati bilo koliko članova, no rade se odsjecanja

(a) TROKUTNO ODSJECANJE

$N(m)=M$

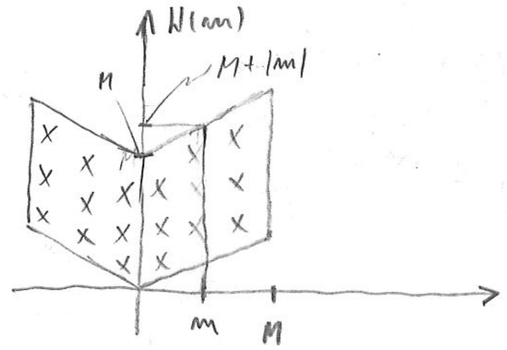
npr. za $M=40 \rightarrow \boxed{T40} \rightarrow$ oznaka modela

Europski centar: $M=235 \rightarrow T235$




(b) ROMBOIDNO ODSJECANJE

$N(m)=M+|m|$

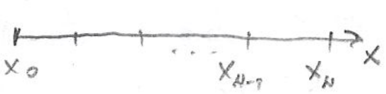



## V.2.1.2.3. METODA KONAČNIH ELEMENATA

- koristi se za približno rješavanje diff. jdžbi pomoću redova s tim da su fje razvoja $\neq 0$ samo na dijelovima područja definicije (domene)
- još se te fje razvoja nazivaju KROV (ROOF) ili ŠEŠIR (HAT), a katkad samo FJE OBLIKA
- taj naziv potječe od podjele kontinuuma na konačne elemente

① FUNKCIJE RAZVOJA

- pp fju $\phi(x,t)$ na području definicije $x\in[x_0,x_N]$ i podijelimo taj interval na N dijelova: $x_{j+1}=x_j+\Delta x$ ; $j=0,1,\dots,N-1$ ;




$x_j$... točke mreže

- $\phi$ se može na svakom lokalnom elementu aproksimirati jdžbom pravca:

$\bar{\phi}(x,t)=\alpha_1(t)+\alpha_2(t)x$ na $\{x_j,x_{j+1}\}$

- sada ću odrediti $\alpha$:

$$\phi_j(t) = \alpha_1(t) + \alpha_2(t)\,x_j$$
$$\phi_{j+1}(t) = \alpha_1(t) + \alpha_2(t)\,x_{j+1} \quad -$$

$$\phi_{j+1}(t) - \phi_j(t) = \alpha_2(t)\left[x_{j+1} - x_j\right] \Rightarrow \boxed{\alpha_2(t) = \frac{\phi_{j+1}(t) - \phi_j(t)}{x_{j+1} - x_j}}$$

$$\alpha_1(t) = \phi_j(t) - \alpha_2(t)\,x_j = \phi_j(t) - \frac{\phi_{j+1}(t) - \phi_j(t)}{x_{j+1} - x_j}\,x_j = \frac{x_{j+1}\phi_j(t) - \cancel{x_j\phi_j(t)} - x_j\phi_{j+1}(t) + \cancel{x_j\phi_j(t)}}{x_{j+1} - x_j}$$

$$\Rightarrow \boxed{\alpha_1(t) = \frac{\phi_j(t)\,x_{j+1} - \phi_{j+1}(t)\,x_j}{x_{j+1} - x_j}}$$

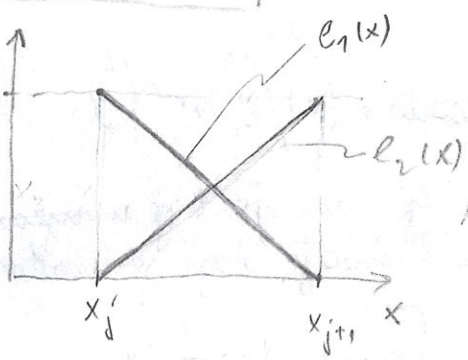
$\phi_j(t)$ ... dobivamo mjerenjem ili objektivnom analizom

- sada:
$$\Phi(x,t) = \frac{\phi_j(t)\,x_{j+1} - \phi_{j+1}(t)\,x_j}{x_{j+1} - x_j} + \frac{\phi_{j+1}(t) - \phi_j(t)}{x_{j+1} - x_j}\,x =$$
$$= \frac{\phi_j(t)\,x_{j+1} - \phi_{j+1}(t)\,x_j + \phi_{j+1}(t)\,x - \phi_j(t)\,x}{x_{j+1} - x_j} = \frac{\phi_j(t)\left[x_{j+1} - x\right] + \phi_{j+1}(t)\left[x - x_j\right]}{x_{j+1} - x_j}$$

$$\Rightarrow \Phi(x,t) = \phi_j(t)\,\frac{x_{j+1} - x}{x_{j+1} - x_j} + \phi_{j+1}(t)\,\frac{x - x_j}{x_{j+1} - x_j} = \phi_j(t)\,e_1(x) + \phi_{j+1}(t)\,e_2(x)$$

$$\Rightarrow \boxed{\Phi(x,t) = \phi_j(t)\,e_1(x) + \phi_{j+1}(t)\,e_2(x) \;\Big|\; e_1(x) = \frac{x_{j+1} - x}{x_{j+1} - x_j} \;\Big|\; e_2(x) = \frac{x - x_j}{x_{j+1} - x_j}}$$

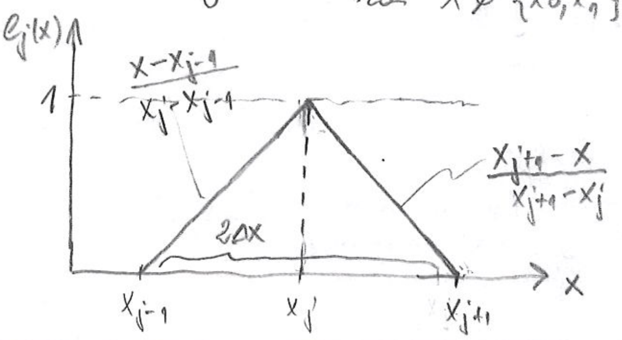
$$\boxed{j = 0, 1, \ldots, N-1 \;\Big|\; x \in [x_j, x_{j+1}]}$$




- mogu zadati opći oblik fja razvoja $e_j$

$$e_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}} & \text{za } x \in \{x_{j-1}, x_j\} \\[2mm] \dfrac{x_{j+1} - x}{x_{j+1} - x_j} & \text{za } x \in \{x_j, x_{j+1}\} \\[2mm] 0 & \text{za } x \notin \{x_{j-1}, x_{j+1}\} \end{cases}$$

$$e_0(x) = \begin{cases} \dfrac{x_1 - x}{x_1 - x_0} & \text{za } x \in \{x_0, x_1\} \\[2mm] 0 & \text{za } x \notin \{x_0, x_1\} \end{cases} \qquad e_N(x) = \begin{cases} \dfrac{x - x_{N-1}}{x_N - x_{N-1}} & \text{za } x \in \{x_{N-1}, x_N\} \\[2mm] 0, & \text{za } x \notin \{x_{N-1}, x_N\} \end{cases}$$

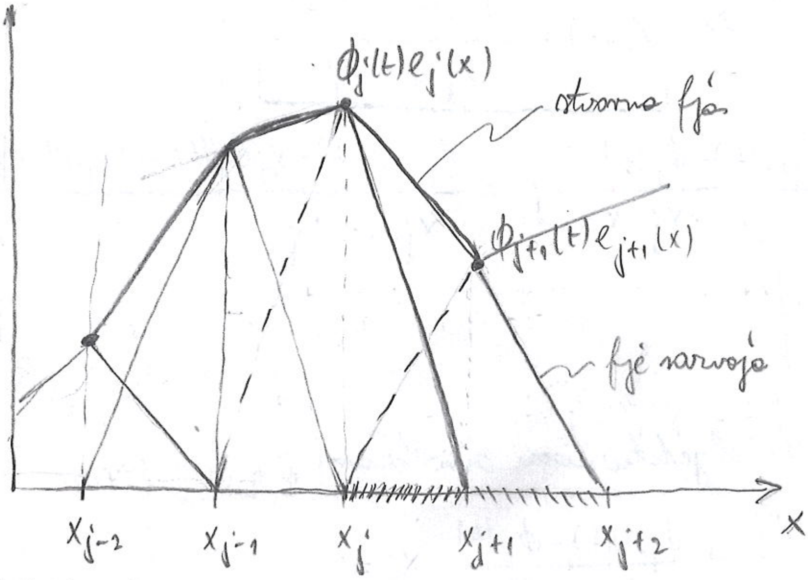



- fje $x_j$ su linearno nezavisne

- može se napraviti općeniti razvoj na cijelom području definicije

$$\Phi(x,t) = \sum_{j=0}^{N} \phi_j(t)\, e_j(x) \quad ; \quad x \in [0,L]$$

- ovo je OK jer smo opet linearizirali problem




## ② PRIMJENA NA LINEARNU ADV. JDŽBU

- znamo otprije: $\frac{\partial \phi(x,t)}{\partial t} + c\,\frac{\partial \phi(x,t)}{\partial x} = 0$
- uvrstimo naš pretpostavljeni razvoj u jdžbu:

$$\boxed{\sum_{j=0}^{N} \frac{d\phi_j(t)}{dt} e_j(x) + c\sum_{j=0}^{N} \phi_j(t)\frac{de_j(x)}{dx} = R \neq 0} \qquad (*)$$

R ... pogreška odsijecanja ⇒ rezidual
- ako ~~pp~~ da R=0 ⇒ loš rezultat ⇒ radije ~~pp~~ R≠0 i primijenimo Galerkinov princip: $\int_0^L R\, e_i(x)\,dx = 0$ , $i \in [0,N]$

$$\Rightarrow \text{sada}: (*) / e_i(x) / \int_0^L dx \;\Rightarrow\; \overbrace{\sum_{j=0}^{N} \frac{d\phi_j(t)}{dt}\int_0^L e_j(x)e_i(x)\,dx}^{\boxed{\mathrm{I}}} + \overbrace{c\sum_{j=0}^{N} \phi_j(t)\int_0^L \frac{de_j(x)}{dx} e_i(x)\,dx}^{\boxed{\mathrm{II}}} = 0$$

- ove fje razvoja koje su ujedno i podintegralne fje imaju svojstvo da njihove kombinacije daju jednostavne rezultate na lokalnim elementima:

$$\int^{x_{j+1}} e_{j\pm1} e_j\,dx = \frac{1}{6}\Delta x \qquad \int \frac{de_j}{dx} e_j\,dx = 0$$

$$\int \frac{de_{j\pm1}}{dx} e_j\,dx = \pm\frac{1}{2}$$

$$\left.\begin{aligned} &\int e_{j\pm p}\, e_j\,dx = 0 \\ &\int \frac{de_{j\pm p}}{dx} e_j\,dx = 0 \end{aligned}\right\} \text{za } p>1$$

$$\int e_j^2\,dx = \frac{2}{3}\Delta x$$

- pokažimo da to vrijedi na primjeru

$$\int_x e_{j+1}e_j\,dx = \int_{x_j}^{x_{j+1}} \left(\frac{x_{j+1}-x}{\underbrace{x_{j+1}-x_j}_{\Delta x}}\right)\left(\frac{x-x_j}{\underbrace{x_{j+1}-x_j}_{\Delta x}}\right)dx = \frac{1}{(\Delta x)^2}\int_{x_j}^{x_{j+1}} (x_{j+1}x - x_j x_{j+1} - x^2 + x_j x)\,dx =$$

$$= \frac{1}{(\Delta x)^2}\left[x_{j+1}\int_{x_j}^{x_{j+1}} x\,dx - x_j x_{j+1}\int_{x_j}^{x_{j+1}} dx - \int_{x_j}^{x_{j+1}} x^2 dx + x_j\int_{x_j}^{x_{j+1}} x\,dx\right] =$$

$$= \frac{1}{(\Delta x)^2}\left[(x_{j+1}+x_j)\cdot\frac{1}{2}(x_{j+1}^2 - x_j^2) - x_j x_{j+1}(x_{j+1}-x_j) - \frac{1}{3}(x_{j+1}^3 - x_j^3)\right] =$$

$$= \frac{1}{(\Delta x)^2}\left[\frac{1}{2}(x_{j+1}^3 - x_j^2 x_{j+1} + x_j x_{j+1}^2 - x_j^3) - x_j x_{j+1}^2 + x_j^2 x_{j+1} - \frac{1}{3}x_{j+1}^3 + \frac{1}{3}x_j^3\right] =$$

$$= \frac{1}{(\Delta x)^2}\left[\frac{1}{2}x_{j+1}^3 - \frac{1}{2}x_j^2 x_{j+1} + \frac{1}{2}x_j x_{j+1}^2 - \frac{1}{2}x_j^3 - x_j x_{j+1}^2 + x_j^2 x_{j+1} - \frac{1}{3}x_{j+1}^3 + \frac{1}{3}x_j^3\right] =$$

$$= \frac{1}{(\Delta x)^2}\left[\frac{1}{6}x_{j+1}^3 + \frac{1}{2}x_j^2 x_{j+1} - \frac{1}{2}x_j x_{j+1}^2 - \frac{1}{6}x_j^3\right] = \frac{1}{6(\Delta x)^2}\underbrace{\left(x_{j+1}^3 + 3x_j^2 x_{j+1} - 3x_j x_{j+1}^2 - x_j^3\right)}_{\text{kub binoma}}$$

$$\Rightarrow \int e_{j+1} e_j\,dx = \frac{1}{6(\Delta x)^2}\cdot\overbrace{(x_{j+1}-x_j)}^{\Delta x}{}^3 = \frac{1}{6}\Delta x \;\; \text{// ok}$$

- [I] u prvom članu sve članove kada je $j = i \pm 1$ te $j = i$ (umjesto $i$ pišem $j$):

$$\frac{d\phi_{j-1}(t)}{dt}\cdot\frac{1}{6}\Delta x + \frac{d\phi_j(t)}{dt}\cdot\frac{2}{3}\Delta x + \frac{d\phi_{j+1}(t)}{dt}\cdot\frac{1}{6}\Delta x \; +$$

- [II] u drugom članu sve članove kada je $j = i \pm 1$ (umjesto $i$ pišem $j$):

$$+\, c\left[\phi_{j-1}(t)\left(-\frac{1}{2}\right) + \phi_{j+1}(t)\cdot\frac{1}{2}\right] = 0$$

$$\Rightarrow \frac{1}{6}\Delta x\left[\frac{d\phi_{j+1}(t)}{dt} + 4\frac{d\phi_j(t)}{dt} + \frac{d\phi_{j-1}(t)}{dt}\right] + \frac{c}{2}\left[\phi_{j+1}(t) - \phi_{j-1}(t)\right] = 0 \;\Big/\; \frac{1}{\Delta x}$$

$$\Rightarrow \boxed{\frac{1}{6}\left(\frac{d\phi_{j+1}}{dt} + 4\frac{d\phi_j}{dt} + \frac{d\phi_{j-1}}{dt}\right) + c\,\frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x} = 0}$$

- sada radim supstituciju: $\frac{d\phi_j^n}{dt} = F_j^n \rightsquigarrow$ vremenski korak; $n = 0, 1, 2, \ldots$

$$\Rightarrow \frac{1}{6}\left(F_{j+1}^n + 4F_j^n + F_{j-1}^n\right) = -c\,\frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x}$$

$$\frac{d\phi_j^n}{dt} = F_j^n = \frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} \quad \ldots \text{centralna shema}$$

- rezultati su slični onima metode konačnih razlika 4. reda
- ova metoda se ne koristi često u meteorologiji