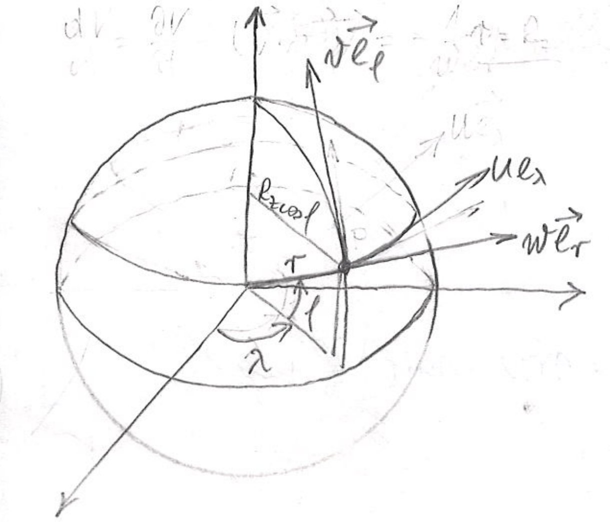
III) HD-JDŽBE U SFERNIM KOORDINATAMA

1) JDŽBA GIBANJA



$$\frac{d\vec V}{dt} = -\frac{1}{\rho}\nabla p - 2\vec\Omega\times\vec V + \vec g + \vec a_{tr}$$

$$\vec V = u\vec e_\lambda + v\vec e_\varphi + w\vec e_r$$

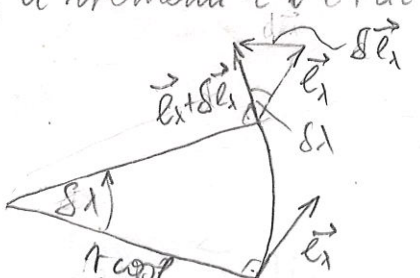
$$\frac{d\vec V}{dt} = \frac{d}{dt}\left(u\vec e_\lambda + v\vec e_\varphi + w\vec e_r\right) =$$

$$= \frac{du}{dt}\vec e_\lambda + u\frac{d\vec e_\lambda}{dt} + \frac{dv}{dt}\vec e_\varphi + v\frac{d\vec e_\varphi}{dt} + \frac{dw}{dt}\vec e_r + w\frac{d\vec e_r}{dt}$$

- trebaju nam vremenske promjene jed. vektora

(a) jedinični vektor $\vec e_\lambda$

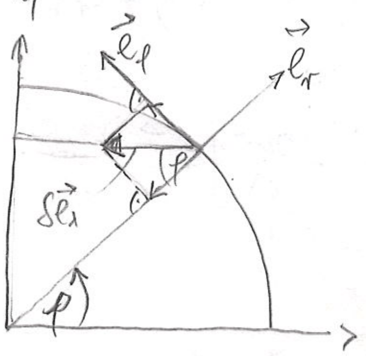
- zamislimo da promatramo duž jedne paralele (u hor. ravnini) jed. vektor $\vec e_\lambda$ u vremenu t i $t+dt$



$$\frac{d\vec e_\lambda}{dt} = \underbrace{\frac{\partial\vec e_\lambda}{\partial t}}_{0} + \frac{u}{r\cos\varphi}\frac{\partial\vec e_\lambda}{\partial\lambda} + \frac{v}{r}\underbrace{\frac{\partial\vec e_\lambda}{\partial\varphi}}_{0} + w\underbrace{\frac{\partial\vec e_\lambda}{\partial r}}_{0}$$

$$\left|\frac{\partial\vec e_\lambda}{\partial\lambda}\right| = \lim_{\delta\lambda\to 0}\frac{|\delta\vec e_\lambda|}{\delta\lambda} = \frac{|\vec e_\lambda|\,\delta\lambda}{\delta\lambda} = 1$$

- smjer promjene $\vec e_\lambda$ je prema središtu rotacije



$$\frac{\partial\vec e_\lambda}{\partial\lambda} = \underbrace{\left|\frac{\partial\vec e_\lambda}{\partial\lambda}\right|}_{1}\left(-\cos\varphi\,\vec e_r + \sin\varphi\,\vec e_\varphi\right)$$

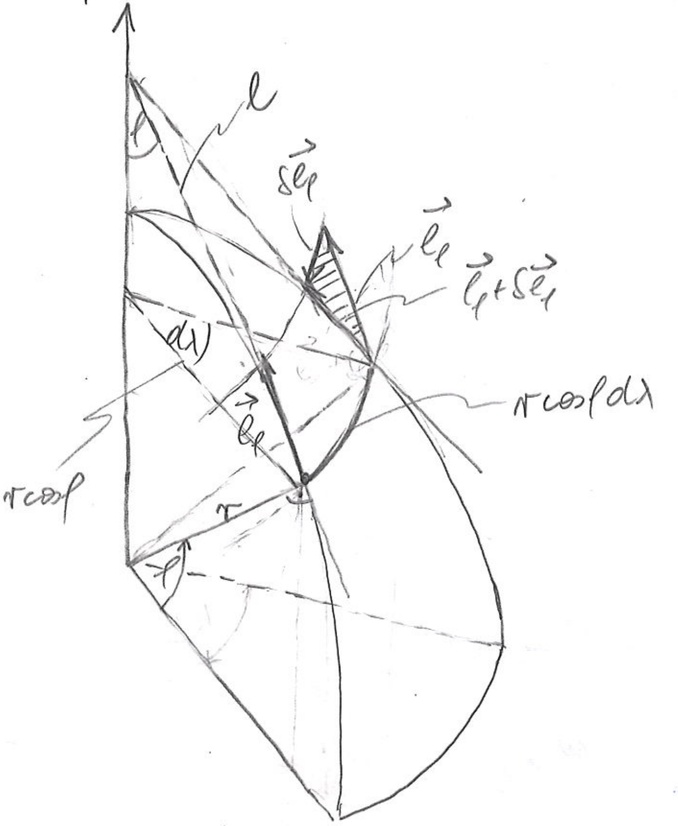
$$\Rightarrow \frac{\partial\vec e_\lambda}{\partial\lambda} = \sin\varphi\,\vec e_\varphi - \cos\varphi\,\vec e_r$$

$$\Rightarrow \boxed{\frac{d\vec e_\lambda}{dt} = \frac{u}{r\cos\varphi}\left(\sin\varphi\,\vec e_\varphi - \cos\varphi\,\vec e_r\right)}$$

(b) jedinični vektor $\vec{e}_\varphi$

$$\frac{d\vec{e}_\varphi}{dt} = \underbrace{\frac{\partial \vec{e}_\varphi}{\partial t}}_{0} + \frac{u}{r\cos\varphi}\frac{\partial \vec{e}_\varphi}{\partial \lambda} + \frac{v}{r}\frac{\partial \vec{e}_\varphi}{\partial \varphi} + w\underbrace{\frac{\partial \vec{e}_\varphi}{\partial r}}_{0}$$

- promjena $\vec{e}_\varphi$-a po λ



$\sin\varphi = \frac{r\cos\varphi}{l} \Rightarrow l = r\,ctg\varphi$

- sličnost trokuta:

$$\frac{r\cos\varphi\, d\lambda}{l} = \frac{|\delta\vec{e}_\varphi|}{|\vec{e}_\varphi|}$$

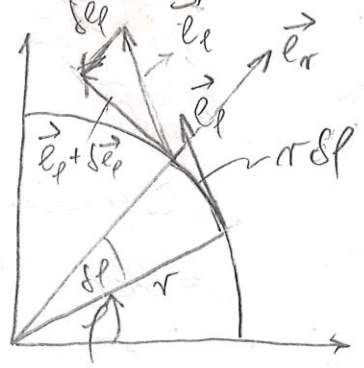
$$\Rightarrow |\delta\vec{e}_\varphi| = \frac{r\cos\varphi\,\delta\lambda}{r\,\frac{\cos\varphi}{\sin\varphi}} = \sin\varphi\,\delta\lambda \;/ : d\lambda$$

$$\Rightarrow \frac{|\delta\vec{e}_\varphi|}{\delta\lambda} = \sin\varphi$$

- promjena vektora $\delta\vec{e}_\varphi$ je u smjeru $(-\vec{e}_\lambda)$

$$\Rightarrow \boxed{\frac{\partial\vec{e}_\varphi}{\partial\lambda} = -\sin\varphi\,\vec{e}_\lambda}$$

- promjena $\vec{e}_\varphi$-a po φ:



- sličnost trokuta: $\frac{|\delta\vec{e}_\varphi|}{|\vec{e}_\varphi|} = \frac{r\,\delta\varphi}{r}$

$$\Rightarrow |\delta\vec{e}_\varphi| = \delta\varphi \;/ : \delta\varphi \Rightarrow \frac{|\delta\vec{e}_\varphi|}{\delta\varphi} = 1 \;/\lim_{\delta\varphi\to 0}$$

$$\Rightarrow \frac{|\partial\vec{e}_\varphi|}{\partial\varphi} = 1$$

- promjena $\vec{e}_\varphi$ je u $(-\vec{e}_r)$ smjeru

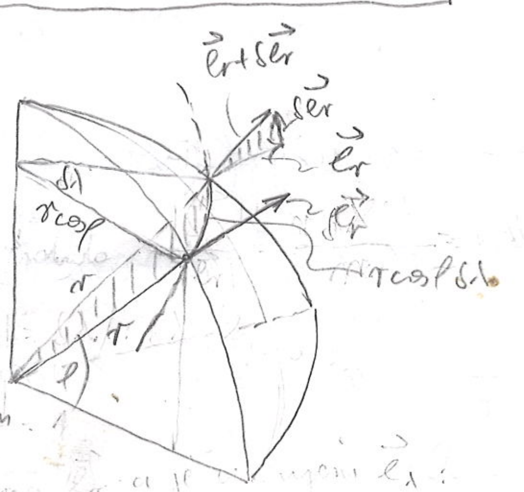
$$\Rightarrow \boxed{\frac{\partial\vec{e}_\varphi}{\partial\varphi} = -\vec{e}_r}$$

tu sam stal 24.10.2018.

$$\Rightarrow \frac{d\vec{e}_\varphi}{dt} = \frac{u}{r\cos\varphi}(-\sin\varphi\,\vec{e}_\lambda) + \frac{v}{r}(-\vec{e}_r) \Rightarrow \boxed{\frac{d\vec{e}_\varphi}{dt} = -u\frac{tg\varphi}{r}\vec{e}_\lambda - \frac{v}{r}\vec{e}_r}$$

(c) jedinični vektor $\vec{e}_r$

$$\frac{d\vec{e}_r}{dt} = \underbrace{\frac{\partial \vec{e}_r}{\partial t}}_{0} + \frac{u}{r\cos\varphi}\frac{\partial \vec{e}_r}{\partial \lambda} + \frac{v}{r}\frac{\partial \vec{e}_r}{\partial \varphi} + w\underbrace{\frac{\partial \vec{e}_r}{\partial r}}_{0}$$



- promjena $\vec{e}_r$-a po λ:

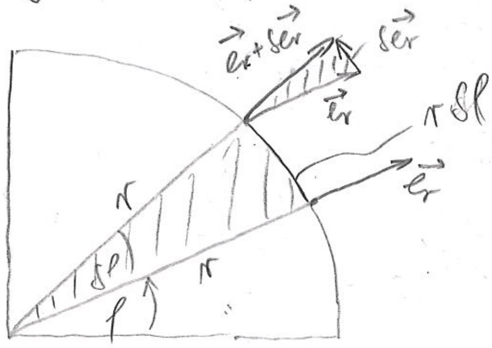
- sličnost trokuta:

$$\frac{|\delta\vec{e}_r|}{\underbrace{|\vec{e}_r|}_{1}} = \frac{r\cos\varphi\,\delta\lambda}{r} = \cos\varphi\,\delta\lambda = |\delta\vec{e}_r| \;/ : \delta\lambda \;/\lim$$

$$\Rightarrow \frac{|\partial\vec{e}_r|}{\partial\lambda} = \cos\varphi$$

- promjena $\vec{e}_r$-a je $\vec{e}_\lambda$ smjeru $\Rightarrow \frac{\partial \vec{e}_r}{\partial \lambda} = \cos\varphi \, \vec{e}_\lambda$

- promjena $\vec{e}_r$-a po φ:



- sličnost Δ: $\frac{|\delta \vec{e}_r|}{|\vec{e}_r|} = \frac{r\,\delta\varphi}{r} = \delta\varphi \;/ : \delta\varphi \;/ \lim$ (uz $|\vec{e}_r| = 1$)

$\Rightarrow \frac{|\partial \vec{e}_r|}{\partial \varphi} = 1$

- promjena $\vec{e}_r$-a je u $\vec{e}_\varphi$ smjeru

$\Rightarrow \frac{\partial \vec{e}_r}{\partial \varphi} = \vec{e}_\varphi$

$$\Rightarrow \frac{d\vec{e}_r}{dt} = \frac{u}{r\cos\varphi}\cos\varphi \, \vec{e}_\lambda + \frac{v}{r}\vec{e}_\varphi \Rightarrow \boxed{\frac{d\vec{e}_r}{dt} = \frac{u}{r}\vec{e}_\lambda + \frac{v}{r}\vec{e}_\varphi}$$

- sada sve to uvrstimo u $\frac{d\vec{V}}{dt}$:

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}\vec{e}_\lambda + u\left[\frac{u}{r\cos\varphi}\left(\sin\varphi\,\vec{e}_\varphi - \cos\varphi\,\vec{e}_r\right)\right] + \frac{dv}{dt}\vec{e}_\varphi + v\left[-u\frac{\mathrm{tg}\,\varphi}{r}\vec{e}_\lambda - \frac{v}{r}\vec{e}_r\right] +$$

$$+ \frac{dw}{dt}\vec{e}_r + w\left[\frac{u}{r}\vec{e}_\lambda + \frac{v}{r}\vec{e}_\varphi\right] = \frac{du}{dt}\vec{e}_\lambda + \frac{u^2}{r\cos\varphi}\sin\varphi\,\vec{e}_\varphi - \frac{u^2}{r\cos\varphi}\cos\varphi\,\vec{e}_r +$$

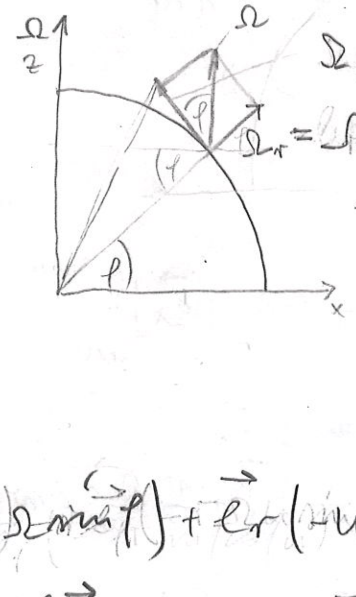
$$+ \frac{dv}{dt}\vec{e}_\varphi - uv\frac{\mathrm{tg}\,\varphi}{r}\vec{e}_\lambda - \frac{v^2}{r}\vec{e}_r + \frac{dw}{dt}\vec{e}_r + \frac{uw}{r}\vec{e}_\lambda + \frac{vw}{r}\vec{e}_\varphi$$

$$\boxed{\frac{d\vec{V}}{dt} = \left(\frac{du}{dt} - uv\frac{\mathrm{tg}\,\varphi}{r} + \frac{uw}{r}\right)\vec{e}_\lambda + \left(\frac{dv}{dt} + \frac{u^2}{r}\mathrm{tg}\,\varphi + \frac{vw}{r}\right)\vec{e}_\varphi + \left(\frac{dw}{dt} - \frac{u^2+v^2}{r}\right)\vec{e}_r}$$

↳ akceleracija u sfernim koord.

- to sada treba vjednačiti sa silama koje uzrokuju tu akceleraciju:

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} + \vec{g} + \vec{a}_{tr}$$



$\Omega_\varphi = \Omega\cos\varphi$

$\Omega_r = \Omega\sin\varphi$

$\Omega_\lambda = 0$

$$\nabla p = \frac{1}{r\cos\varphi}\frac{\partial p}{\partial \lambda}\vec{e}_\lambda + \frac{1}{r}\frac{\partial p}{\partial \varphi}\vec{e}_\varphi + \frac{\partial p}{\partial r}\vec{e}_r$$

$$\vec{\Omega}\times\vec{V} = \begin{vmatrix} \vec{e}_\lambda & \vec{e}_\varphi & \vec{e}_r \\ 0 & \Omega\cos\varphi & \Omega\sin\varphi \\ u & v & w \end{vmatrix} =$$

$$= \vec{e}_\lambda\left(w\Omega\cos\varphi - v\Omega\sin\varphi\right) - \vec{e}_\varphi\left(-u\Omega\sin\varphi\right) + \vec{e}_r\left(-u\Omega\cos\varphi\right) =$$

$$= \Omega\left(w\cos\varphi - v\sin\varphi\right)\vec{e}_\lambda + \Omega u\sin\varphi\,\vec{e}_\varphi - \Omega u\cos\varphi\,\vec{e}_r$$

$$-2\vec{\Omega}\times\vec{V} = -2\Omega(w\cos\varphi - v\sin\varphi)\vec{e}_\lambda - 2\Omega u\sin\varphi\,\vec{e}_\varphi + 2\Omega u\cos\varphi\,\vec{e}_r$$
$$= (fv - 2w\Omega\cos\varphi)\vec{e}_\lambda - fu\,\vec{e}_\varphi + 2u\Omega\cos\varphi\,\vec{e}_r$$

$\vec{g} = -g\vec{e}_r$; $\vec{a}_{tr} = a_\lambda\vec{e}_\lambda + a_\varphi\vec{e}_\varphi + a_r\vec{e}_r$

- rastav jdžbe gibanja po komponentama:

$$\lambda \ldots \quad \frac{du}{dt} - \underbrace{uv\frac{\operatorname{tg}\varphi}{r} + \frac{uw}{r}}_{\ll} = -\frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial\lambda} + fv - \underbrace{2w\Omega\cos\varphi}_{\ll} + a_\lambda$$

$$\varphi \ldots \quad \frac{dv}{dt} + \frac{u^2}{r}\operatorname{tg}\varphi + \underbrace{\frac{vw}{r}}_{\ll} = -\frac{1}{\rho r}\frac{\partial p}{\partial\varphi} - fu + a_\varphi$$

$$r \ldots \quad \frac{dw}{dt} - \underbrace{\frac{u^2+v^2}{r}}_{\ll} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \underbrace{2u\Omega\cos\varphi}_{\ll} - g + a_r$$

→ OSNOVNE JDŽBE (nakon opisat... primitivne)

② JDŽBA KONTINUITETA

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\vec{V}) \Rightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0 \Rightarrow \frac{\partial\rho}{\partial t} + \rho\nabla\cdot\vec{V} + \vec{V}\cdot\nabla\rho = 0 \Rightarrow \frac{d\rho}{dt} + \rho\nabla\cdot\vec{V} = 0 \;/\cdot\frac{1}{\rho}$$

$$\Rightarrow \boxed{\frac{1}{\rho}\frac{d\rho}{dt} + \nabla\cdot\vec{V} = 0}$$

$$\Rightarrow \frac{1}{\rho}\left(\frac{\partial\rho}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial\rho}{\partial\lambda} + \frac{v}{r}\frac{\partial\rho}{\partial\varphi} + w\frac{\partial\rho}{\partial r}\right) + \frac{1}{r\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{1}{r\cos\varphi}\frac{\partial(\cos\varphi\, v)}{\partial\varphi} + \frac{1}{r^2}\frac{\partial(r^2 w)}{\partial r} = 0$$

③ I ZTD-a

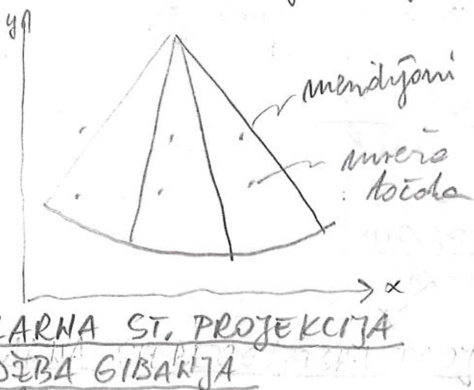
$$Q = c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = c_v\left(\frac{\partial T}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial T}{\partial\lambda} + \frac{v}{r}\frac{\partial T}{\partial\varphi} + w\frac{\partial T}{\partial r}\right) + p\left(\frac{\partial\alpha}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial\alpha}{\partial\lambda} + \frac{v}{r}\frac{\partial\alpha}{\partial\varphi} + w\frac{\partial\alpha}{\partial r}\right)$$

④ JDŽBA STANJA IP-a

$$p\alpha = RT$$

Ⓘⓥ JDŽBE U SUSTAVU KARTE

- često se sfera ili njeni dijelovi projeciraju na pravokutni sustav
- prije integracije, jdžbe treba prebaciti u željeni sustav



POLARNA ST. PROJEKCIJA

① JDŽBA GIBANJA

- u sfernom sustavu elementi pomaka su:

$$ds_1 = h_1 dx_1 \Rightarrow ds_1 = h_\lambda d\lambda = r\cos\varphi\, d\lambda \;/\cdot\frac{1}{dt} \Rightarrow \frac{ds_1}{dt} = u = r\cos\varphi\frac{d\lambda}{dt}$$

$$ds_2 = h_2 dx_2 \Rightarrow ds_2 = h_\varphi d\varphi = r\, d\varphi \;/\cdot\frac{1}{dt} \Rightarrow \frac{ds_2}{dt} = v = r\frac{d\varphi}{dt}$$

$$ds_3 = h_3 dx_3 \Rightarrow ds_3 = h_r dr = dr \;/\cdot\frac{1}{dt} \Rightarrow w = \frac{dr}{dt}$$

- dakle: $u = r\cos\varphi \frac{d\lambda}{dt}$; $v = r\frac{d\varphi}{dt}$; $w = \frac{dr}{dt}$

- to sada uvrstimo u komponente jdbe gibanja

$$\lambda \ldots \quad \frac{d}{dt}(r\cos\varphi\dot\lambda) - r\cos\varphi\dot\lambda r\dot\varphi\frac{tg\varphi}{r} + r\cos\varphi\dot\lambda\dot r\frac{1}{r} = 2\Omega\sin\varphi r\dot\varphi - 2\Omega\cos\varphi\dot r + \underbrace{-\frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial\lambda} + a_\lambda}_{F_\lambda'}$$

$$\Rightarrow \dot r\cos\varphi\dot\lambda + r\dot\lambda(-\sin\varphi)\dot\varphi + r\cos\varphi\ddot\lambda - r\sin\varphi\dot\lambda\dot\varphi + \cos\varphi\dot\lambda\dot r - 2\Omega\sin\varphi r\dot\varphi + 2\Omega\cos\varphi\dot r = F_\lambda'$$

$$\Rightarrow r\cos\varphi\ddot\lambda + 2\cos\varphi\dot r\dot\lambda - 2r\sin\varphi\dot\lambda\dot\varphi - 2\Omega\sin\varphi r\dot\varphi + 2\Omega\cos\varphi\dot r = F_\lambda'$$

$$\Rightarrow r\cos\varphi\ddot\lambda + 2\cos\varphi\dot r(\dot\lambda+\Omega) - 2r\sin\varphi\dot\varphi(\dot\lambda+\Omega) = F_\lambda'$$

$$\boxed{\lambda \ldots \quad r\cos\varphi\ddot\lambda + 2(\dot\lambda+\Omega)(\dot r\cos\varphi - r\sin\varphi\dot\varphi) = F_\lambda'}$$

$$\varphi \ldots \quad \frac{d}{dt}(r\dot\varphi) + \frac{tg\varphi}{r}r^2\cos^2\varphi\dot\lambda^2 + \frac{1}{r}r\dot\varphi\dot r + 2\Omega\sin\varphi r\cos\varphi\dot\lambda = -\frac{1}{\rho r}\frac{\partial p}{\partial\varphi} + a_\varphi = F_\varphi'$$

$$\Rightarrow \dot r\dot\varphi + r\ddot\varphi + \sin\varphi\cos\varphi r\dot\lambda^2 + \dot\varphi\dot r + 2\Omega\sin\varphi r\cos\varphi\dot\lambda = F_\varphi'$$

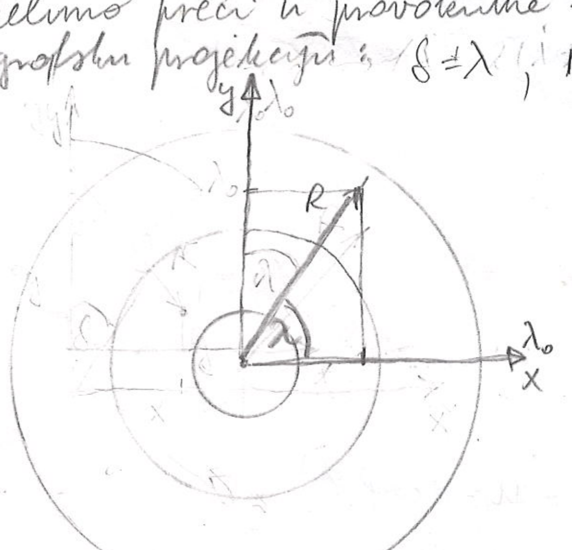
$$\boxed{\varphi \ldots \quad r\ddot\varphi + 2\dot r\dot\varphi + r\dot\lambda\sin\varphi\cos\varphi(\dot\lambda+2\Omega) = F_\varphi'}$$

$$r \ldots \quad \frac{d}{dt}\dot r - \frac{1}{r}(r^2\cos^2\varphi\dot\lambda^2 + r^2\dot\varphi^2) - 2\Omega\cos\varphi r\cos\varphi\dot\lambda = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + a_r = F_r'$$

$$\Rightarrow \ddot r - r\cos^2\varphi\dot\lambda^2 - r\dot\varphi^2 - 2\Omega r\cos^2\varphi\dot\lambda = F_r'$$

$$\boxed{r \ldots \quad \ddot r - r\dot\varphi^2 - r\cos^2\varphi\dot\lambda(\dot\lambda+2\Omega) = F_r'}$$

- želimo preći u pravokutne koordinate ⇒ uzmimo da je za polarnu stereografsku projekciju: $\delta = \lambda$; $R = 2a\frac{\cos\varphi}{1+\sin\varphi}$; a... polumjer Z



$$x = R\cos\lambda = 2a\frac{\cos\varphi\cos\lambda}{1+\sin\varphi}$$

$$y = R\sin\lambda = 2a\frac{\cos\varphi\sin\lambda}{1+\sin\varphi}$$

$$z = r - a$$

- brzine u sustavu karte:

$U = h_x\dot x$; $V = h_y\dot y$; x, y... generalizirane koordinate

- brzine na sferi (krivolinijske koordinate):

$u = \frac{ds_1}{dt}$; $v = \frac{ds_2}{dt}$; s_1, s_2 ... dijelovi luka

- promotrimo odnos duljinskih diferencijala na karti i na sferi:

<u>karta</u> (2) $dy = 2a \frac{(-\sin\varphi d\varphi)\sin\lambda + \cos\varphi(\cos\lambda d\lambda)(1+\sin\varphi) - \cos\varphi\sin\lambda\cos\varphi d\varphi}{(1+\sin\varphi)^2} =$

$= 2a \frac{-\sin\varphi\sin\lambda d\varphi + \cos\varphi\cos\lambda d\lambda - \sin^2\varphi\sin\lambda d\varphi + \sin\varphi\cos\varphi\cos\lambda d\lambda -}{(1+\sin\varphi)^2} \ldots$

$\ldots \frac{-\cos^2\varphi\sin\lambda d\varphi}{} =$

$= 2a \frac{-\sin\varphi\sin\lambda d\varphi + \cos\varphi\cos\lambda d\lambda - \sin\lambda d\varphi \underbrace{(\sin^2\varphi + \cos^2\varphi)}_{1} + \sin\varphi\cos\varphi\cos\lambda d\lambda}{(1+\sin\varphi)^2} =$

$= 2a \frac{-\sin\lambda d\varphi (\sin\varphi + 1) + \cos\varphi\cos\lambda d\lambda (1 + \sin\varphi)}{(1+\sin\varphi)^2} =$

$\Rightarrow dy = \frac{2a}{1+\sin\varphi} (\cos\varphi\cos\lambda d\lambda - \sin\lambda d\varphi)$

- analogno ... (1) $dx = \frac{2a}{1+\sin\varphi}(-\cos\varphi\sin\lambda d\lambda - \cos\lambda d\varphi)$

<u>sferni sustav (Zemlja)</u>

(3) $ds_1 = h_1 dx_1 = r\cos\varphi d\lambda$

(4) $ds_2 = h_2 dx_2 = r d\varphi$

$ds_3 = h_3 dx_3 = dr$

(3) $\Rightarrow d\lambda = \frac{ds_1}{r\cos\varphi}$; (4) $\Rightarrow d\varphi = \frac{ds_2}{r}$

(1) $\Rightarrow dx = \frac{2a}{1+\sin\varphi}\left(-\cos\varphi\sin\lambda \frac{ds_1}{r\cos\varphi} - \cos\lambda \frac{ds_2}{r}\right) = \frac{2a}{r(1+\sin\varphi)}(-\sin\lambda ds_1 - \cos\lambda ds_2)$

(2) $\Rightarrow dy = \frac{2a}{1+\sin\varphi}\left(\cos\varphi\cos\lambda \frac{ds_1}{r\cos\varphi} - \sin\lambda \frac{ds_2}{r}\right) = \frac{2a}{r(1+\sin\varphi)}(\cos\lambda ds_1 - \sin\lambda ds_2)$

- matrično zapisano:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \frac{2a}{r(1+\sin\varphi)} \begin{pmatrix} -\sin\lambda & -\cos\lambda \\ \cos\lambda & -\sin\lambda \end{pmatrix} \begin{pmatrix} ds_1 \\ ds_2 \end{pmatrix} \rightsquigarrow (*)$$

metrički koeficijent — matrica rotacije (koliko se moraju duljine sa sfere rotirati da bi ih prikazali na karti)

↳ koliko su se „smanjile" linije kada su se prenijele na kartu

$h_x = h_y = \frac{1}{m} = \frac{r(1+\sin\varphi)}{2a} = \{r \simeq a\} \approx \frac{1+\sin\varphi}{2} = h$

- sada mogu koristiti Laméove koeficijente za brzine u sustavu karte:

$dX = h_x dx$; $dY = h_y dy$ $\Rightarrow$ $U = h\dot{x}$; $V = h\dot{y}$

- zada $\Rightarrow \dot{x} = \frac{1}{h} u \quad ; \quad \dot{y} = \frac{1}{h} v$

- podijelimo (*) sa dt: $\frac{1}{h}\begin{pmatrix} U \\ V \end{pmatrix} = \frac{1}{h}\begin{pmatrix} -\sin\lambda & -\cos\lambda \\ \cos\lambda & -\sin\lambda \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$$\Rightarrow \boxed{\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} -\sin\lambda & -\cos\lambda \\ \cos\lambda & -\sin\lambda \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}}$$