(4) a) $A=\{1+(-1)^n\cdot\frac{1}{n} : n\in\mathbb{N}\}$

$A_1 := \{1+\frac{1}{2n} : n\in\mathbb{N}\}$, $A_2=\{1-\frac{1}{2n-1} : n\in\mathbb{N}\}$

$$A=A_1\cup A_2 \overset{\text{zad 2. a)}}{\Longrightarrow} \mathrm{Cl}\,A=\mathrm{Cl}\,A_1\cup \mathrm{Cl}\,A_2$$

$a_n = 1+\frac{1}{2n}$; $n\in\mathbb{N}$ $(a_n)_n$ je niz u A_1 i $\lim a_n = 1$

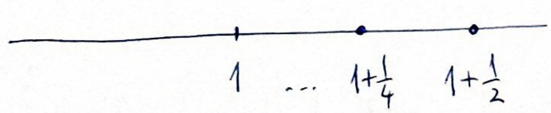
$$\underset{\substack{\text{propozicija}\\ \text{s predavanja}}}{\Longrightarrow} 1\in \mathrm{Cl}\,A_1$$

<u>Tvrdnja</u> $\mathrm{Cl}\,A_1 = A_1\cup\{1\}$

Već smo dokazali $1\in \mathrm{Cl}\,A_1$. $A_1\cup\{1\}\subseteq \mathrm{Cl}\,A_1$ (1)

Da bismo pokazali jednakost skupova, dovoljno je dokazati da je $A_1\cup\{1\}$ zatvoren skup.

$$(A_1\cup\{1\})^c = \langle-\infty,1\rangle\cup\langle\tfrac{3}{2},+\infty\rangle\cup\bigcup_{n\in\mathbb{N}}\langle 1+\tfrac{1}{2n+2},1+\tfrac{1}{2n}\rangle$$



$(A_1\cup\{1\})^c$ je unija otvorenih skupova $\Rightarrow (A_1\cup\{1\})^c$ je otvoren skup $\Rightarrow A_1\cup\{1\}$ je zatvoren.

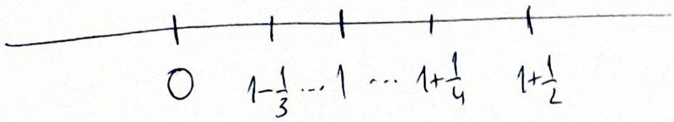
$\Rightarrow \mathrm{Cl}\,A_1\subseteq A_1\cup\{1\}$ (2)

(1) & (2) $\Rightarrow \mathrm{Cl}\,A_1 = A_1\cup\{1\}$

Analogno se dokaže $\mathrm{Cl}\,A_2 = A_2\cup\{1\}$

$$\Rightarrow \boxed{\mathrm{Cl}\,A = A_1\cup\{1\}\cup A_2\cup\{1\} = A\cup\{1\}}$$

Za $x \in A \Rightarrow x = 1 + (-1)^n \cdot \frac{1}{n}$ za neki $n \in \mathbb{N}$



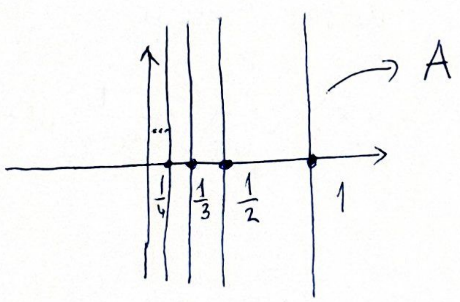
$\forall \varepsilon > 0 \quad K(x, \varepsilon) = \langle x-\varepsilon, x+\varepsilon \rangle \not\subseteq A$ (zato što $\langle x-\varepsilon, x+\varepsilon \rangle$ sadrži i npr. neke iracionalne brojeve, a iracionalni brojevi nisu u A)

$\Rightarrow x \notin \operatorname{Int} A \quad \forall x \in A$

Dakle, $\boxed{\operatorname{Int} A = \emptyset}$

zad 2. c) & d) $\Rightarrow \boxed{\partial A = \operatorname{Cl} A \setminus \operatorname{Int} A = A \cup \{1\}}$

4. b)



Za fiksni $y \in \mathbb{R}$ niz $\left(\left(\frac{1}{n}, y\right)\right)_n$ u $\mathbb{R}^2$ je niz u A čiji je limes $(0, y)$ pa je po propoziciji s predavanja $(0, y) \in \operatorname{Cl} A$. To vrijedi za svaki $y \in \mathbb{R}$

$\Rightarrow A \cup (\{0\} \times \mathbb{R}) \subseteq \operatorname{Cl} A \quad (1)$

Dokažimo da vrijedi i obratna inkluzija.

Tvrdimo da je $A \cup \{0\} \times \mathbb{R}$ zatvoren skup u $\mathbb{R}^2$.

$$(A \cup (\{0\}\times\mathbb{R}))^c = \Big(\langle -\infty, 0\rangle \cup \langle 1, +\infty\rangle \cup \bigcup_{n\in\mathbb{N}} \langle \tfrac{1}{n+1}, \tfrac{1}{n}\rangle\Big)\times \mathbb{R}$$

$$= \underbrace{\langle -\infty, 0\rangle \times \mathbb{R}}_{\text{otvoren}} \cup \underbrace{\langle 1, +\infty\rangle \times \mathbb{R}}_{\text{otvoren}} \bigcup_{n\in\mathbb{N}} \underbrace{\langle \tfrac{1}{n+1}, \tfrac{1}{n}\rangle \times \mathbb{R}}_{\text{otvoren}}$$

$(A \cup (\{0\}\times\mathbb{R}))^c$ je unija otvorenih skupova u $\mathbb{R}^2$ pa je $(A \cup (\{0\}\times\mathbb{R}))^c$ otvoren skup $\Rightarrow$ $A \cup (\{0\}\times\mathbb{R})$ je zatvoren

$\Rightarrow \operatorname{Cl} A \subseteq A \cup (\{0\}\times\mathbb{R})$ (2)

(1) & (2) $\Rightarrow$ $\boxed{\operatorname{Cl} A = A \cup (\{0\}\times\mathbb{R})}$

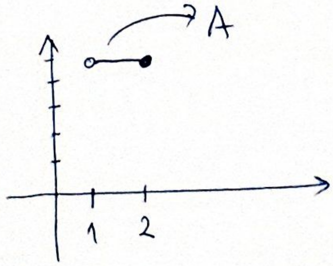
Za $x \in A \Rightarrow x = (\frac{1}{n}, y)$ za neki $n \in \mathbb{N}$ i neki $y \in \mathbb{R}$

$\forall \varepsilon > 0$ $K(x,\varepsilon) \nsubseteq A$ (npr. $K(x,\varepsilon)$ sadrži neke elemente iz skupa $(\mathbb{R}\setminus\mathbb{Q})\times\mathbb{R}$, a ti elementi nisu iz A).

$\Rightarrow x \notin \operatorname{Int} A$ $\forall x \in A$ $\Rightarrow$ $\boxed{\operatorname{Int} A = \emptyset}$

$$\boxed{\partial A = \operatorname{Cl} A \setminus \operatorname{Int} A = A \cup (\{0\}\times\mathbb{R})}$$

4. c)



$(1+\frac{1}{n}, 5)_n$ je niz u A i $\lim_n (1+\frac{1}{n}, 5) = (1,5)$

$\Rightarrow (1,5) \in \operatorname{Cl} A$

$\Rightarrow \operatorname{Cl} A \supseteq A \cup \{(1,5)\} = [1,2]\times\{5\}$ (1)

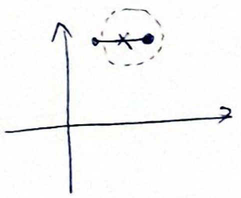
$$([1,2]\times\{5\})^c = \underbrace{(\langle -\infty, 1\rangle \times \mathbb{R})}_{\text{otvoren}} \cup \underbrace{(\langle 2, +\infty\rangle \times \mathbb{R})}_{\text{otvoren}} \cup \underbrace{\mathbb{R}\times\langle -\infty, 5\rangle}_{\text{otvoren}} \cup \underbrace{\mathbb{R}\times\langle 5, +\infty\rangle}_{\text{otvoren}}$$

$\Rightarrow ([1,2]\times\{5\})^c$ je otvoren $\Rightarrow [1,2]\times\{5\}$ je zatvoren u $\mathbb{R}^2$ i sadrži A.

$\Rightarrow \mathrm{Cl}\,A \subseteq [1,2]\times\{5\}$ (2)

(1) & (2) $\Rightarrow \boxed{\mathrm{Cl}\,A = [1,2]\times\{5\}}$

Za $x \in A \Rightarrow x = (x_1, 5),\ x_1 \in \langle 1,2]$



$\forall \varepsilon > 0\ K(x,\varepsilon) \not\subseteq A$ (kugla sadrži elemente čija je druga koordinata $\neq 5$)

$\Rightarrow \boxed{\mathrm{Int}\,A = \emptyset}$

$\boxed{\partial A = \mathrm{Cl}\,A \setminus \mathrm{Int}\,A = [1,2]\times\{5\}}$