Popravni 2016 4 a) Neka je $f: \mathbb{R}^2 \to \mathbb{R}$ diferencijabilna funkcija t.d. za sve $x, y \in \mathbb{R}$ vrijedi $f(x,y) = f(y,x)$. Koristeći definiciju diferencijala dokažite da za sve $x, y \in \mathbb{R}$ vrijedi $Df(x,y) = Df(y,x)^{\ominus}$; pritom za matricu $A = [a, b]$ definiramo $A^{\ominus} := [b, a]$.

Rj: $Df(x,y) = f'(x,y) = \left[\frac{\partial f}{\partial x}(x,y) \quad \frac{\partial f}{\partial y}(x,y) \right]$

$$Df(y,x) = f'(y,x) = \left[\frac{\partial f}{\partial x}(y,x), \quad \frac{\partial f}{\partial y}(y,x) \right]$$

Treba dokazati da je $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(y,x)$ &

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial x}(y,x) \quad \forall (x,y) \in \mathbb{R}^2$$

$$\frac{\partial f}{\partial y}(y,x) = \lim_{h \to 0} \frac{f(y,x+h) - f(y,x)}{h} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \frac{\partial f}{\partial x}(x,y)$$

$$\frac{\partial f}{\partial x}(y,x) = \lim_{h \to 0} \frac{f(y+h,x) - f(y,x)}{h} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$= \frac{\partial f}{\partial y}(x,y)$$

PPZ 2020.

2 a) Neka su $A, B \subseteq \mathbb{R}^n$ neprazni skupovi i stavimo $A+B = \{a+b \in \mathbb{R}^n ;\ a \in A, b \in B\}$. Ako su A i B otvoreni u $\mathbb{R}^n$, onda je i $A+B$ otvoren. Dokažite.

Rj: Neka je $x \in A+B$ proizvoljni element. Tada je $x = a+b$ za neki $a \in A$, $b \in B$.

A je otvoren skup i $a \in A \Rightarrow \exists\, r > 0$ t.d. $K(a,r) \subseteq A$.

Dokažimo da je $K(x,r) \subseteq A+B$

Neka je $y \in K(x,r) \Rightarrow \|x-y\| < r$

$$\Rightarrow \|a+b-y\| < r$$

$$\Rightarrow \|a+(b-y)\| < r$$

$$\Rightarrow \|a-(y-b)\| < r$$

$$\Rightarrow y-b \in K(a,r) \subseteq A$$

$$y = \underbrace{(y-b)}_{\in A} + \underbrace{b}_{\in B} \Rightarrow y \in A+B$$

$\Rightarrow A+B$ je otvoren

PPZ 2020. 4 a) Za funkciju $f: X \to Y$ oblika $f(x) = Ax + y$ za svaki $x \in X$ i neki $y \in Y$, gdje su X i Y normirani prostori i $A: X \to Y$ ograničen linearan operator, provjerite je li diferencijabilna u svakoj točki X. Ako je, za svaki $x \in X$ nađite $f'(x)$.

rj: Budući da nismo ove godine radili operatore na normiranim prostorima i ograničene lin. op, možete slobodno raditi s $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$.

$$f(x) = Ax + y \qquad f: \mathbb{R}^n \to \mathbb{R}^m$$

Što bi mogao biti kandidat za diferencijal?

$$\frac{\| f(x_0+h) - f(x_0) - Th \|}{\|h\|} = \frac{\| (A(x_0+h)+y) - (Ax_0+y) - Th \|}{\|h\|}$$

$$\underbrace{=}_{A \text{ je lin. op.}} \frac{\| \cancel{Ax_0} + Ah + \cancel{y} - \cancel{Ax_0} - \cancel{y} - Th \|}{\|h\|} =$$

$$= \frac{\| Ah - Th \|}{\|h\|}$$

Stavimo li $T = A$, ići će u 0.

$$\Rightarrow \lim_{h \to 0} \frac{\| f(x_0+h) - f(x_0) - Ah \|}{\|h\|} = 0 \Rightarrow f \text{ je dif. u svakoj točki } x \in X \text{ i } f'(x) = A$$

1. KOL. 2021 3 c) Neka su $(x_n)_n$ i $(y_n)_n$ dva ograničena niza u $\mathbb{R}$. Dokažite ili opovrgnite tvrdnju:

$$\limsup_{n\to\infty}(x_n+y_n)=\limsup_{n\to\infty}x_n+\limsup_{n\to\infty}y_n$$

rj: Stavimo $\left.\begin{array}{l} x_n=(-1)^n \\ y_n=-(-1)^n \end{array}\right\} \forall n\in\mathbb{N}$

$x_n+y_n=0 \quad \forall n\in\mathbb{N} \quad \Rightarrow \limsup_{n\to\infty}(x_n+y_n)=0$

$\limsup_{n\to\infty}x_n=\limsup_{n\to\infty}(-1)^n=1$ (jer niz $((-1)^n)_n$ ima dva gomilišta, 1 i -1)

$\limsup_{n\to\infty}y_n=\limsup_{n\to\infty}-(-1)^n=1$ (jer niz $(-(-1)^n)_n$ ima dva gomilišta, 1 i -1)

$0\neq 1+1$

Dakle, tvrdnja općenito ne vrijedi.

1. kol 2022: 3 c) Neka su $(a_n)_n, (b_n)_n$ i $(c_n)_n$ nizovi realnih brojeva t.d. $a_n \leq b_n \leq c_n$ za sve $n \in \mathbb{N}$. Ako $(a_n)_n$ i $(c_n)_n$ imaju isto gomilište, mora li i $(b_n)_n$ imati to gomilište?

rj:

$(a_n)_n$	1	3	1	3	1	3	...
$(b_n)_n$	2	4	2	4	2	4	...
$(c_n)_n$	3	5	3	5	3	5	...

Stavimo

$$a_{2k-1} = 1 \quad \forall k \in \mathbb{N}$$
$$a_{2k} = 3 \quad \forall k \in \mathbb{N}$$

$$b_{2k-1} = 2 \quad \forall k \in \mathbb{N}$$
$$b_{2k} = 4 \quad \forall k \in \mathbb{N}$$

$$c_{2k-1} = 3 \quad \forall k \in \mathbb{N}$$
$$c_{2k} = 5 \quad \forall k \in \mathbb{N}$$

Vrijedi $a_n \leq b_n \leq c_n \ \forall n \in \mathbb{N}$

Nizovi $(a_n)_n$ i $(c_n)_n$ imaju isto gomilište 3, a niz $(b_n)_n$ ima gomilišta 2 i 4, a 3 nije njegovo gomilište

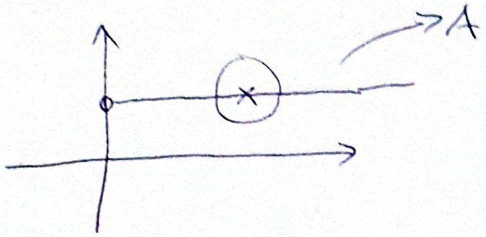
$\Rightarrow$ Ne može

1. KOL. 2022 4 b) Neka je (X, d) metrički prostor i $A \subseteq X$.

Odredite zatvarač i interior skupa

$A = \{(x,1) : x > 0\}$ u $(\mathbb{R}^2, d_2)$

rj:



$\forall\ (x,1) \in A$ će kugla oko $(x,1)$ (bilo kojeg radijusa) r u sebi sadržavati točke koje nisu u A.

Npr. $K((x,1), r)$ sadrži u sebi točku $(x, 1+\frac{r}{2})$, a ta točka nije u A. Dakle, $\text{int}\, A = \emptyset$

$Cl\, A = A \cup A'$

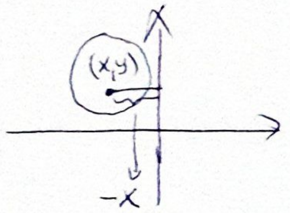
Jedino gomilište skupa A koje nije u A je točka $(0,1)$.

$a_n = (\frac{1}{n}, 1) \in A \qquad \lim\limits_{n\to\infty} a_n = (0,1) \Rightarrow (0,1) \in Cl\, A$

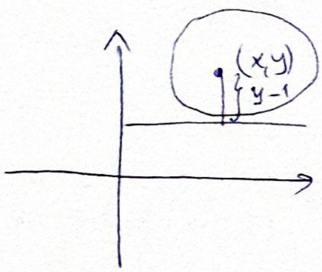
$Cl\, A \supseteq \{(x,1) : x \geq 0\}$ Da bismo dokazali obratnu inkluziju, dovoljno je dokazati da je $\{(x,1) : x \geq 0\}$ zatvoren skup, tj. da je njegov komplement otvoren.

Ako je $(x,y) \in \{(x,1) : x \geq 0\}^c$, onda je ili $x < 0$ ili $x \geq 0$ i $y \neq 1$.

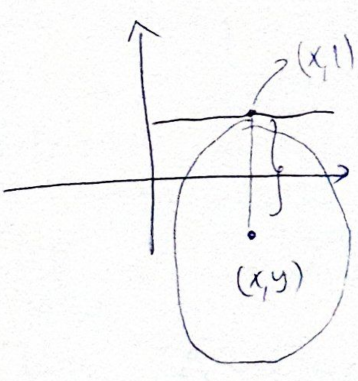
- Ako je $x<0$, onda je $K((x,y),-x) \subseteq \{(x,1) : x \geq 0\}^c$



- Ako je $x \geq 0$ i $y>1$, onda je $K((x,y),y-1) \subseteq \{(x,1); x\geq 0\}^c$



- Ako je $x \geq 0$ i $y<1$, onda je $K((x,y),1-y) \subseteq \{(x,1) : x \geq 0\}^c$

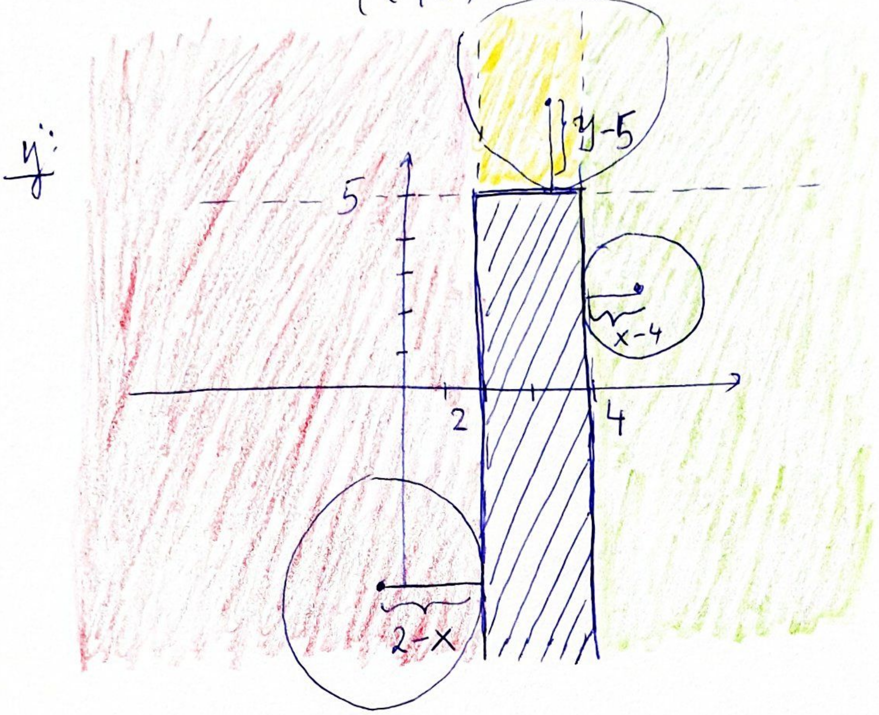


Dakle, $\{(x,1) : x\geq 0\}^c$ je otvoren $\Rightarrow \{(x,1) : x \geq 0\}$ je zatvoren

$Cl\, A = \{(x,1) : x \geq 0\}$

1. KOL. 2022. 4c) Dokažite po definiciji da je skup

$\{(x,y)\in\mathbb{R}^2 : x\in[2,4],\ y\le 5\}$ zatvoren u $\mathbb{R}^2$.



$A = \{(x,y)\in\mathbb{R}^2 : x\in[2,4],\ y\le 5\}$

$A^C = A_1 \cup A_2 \cup A_3$

$A_1 = \{(x,y)\in\mathbb{R}^2 : x<2\}$ (crveni)

$A_2 = \{(x,y)\in\mathbb{R}^2 : x\in[2,4],\ y>5\}$ (žuti)

$A_3 = \{(x,y)\in\mathbb{R}^2 : x>4\}$ (zeleni)

Za $(x,y)\in A_1$ je $K((x,y), 2-x) \subseteq A_1 \subseteq A^C$

Za $(x,y)\in A_2$ je $K((x,y), y-5) \subseteq A^C$

Za $(x,y)\in A_3$ je $K((x,y), x-4) \subseteq A_3 \subseteq A^C$

Dakle, A^C je otvoren $\Rightarrow$ A je zatvoren

1. popravak PPZ 2020 3 a) Dokažite da je preslikavanje

$\pi : \mathbb{R}^2 \to \mathbb{R}^2$, $\pi(x,y) = (x,x)$ uniformno

neprekidno na $\mathbb{R}^2$.

rj: Treba dokazati da $(\forall \varepsilon > 0)$ $(\exists \delta > 0)$ t.d.

$(\forall\ (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2)$ $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} < \delta$ da je

$$d(\pi(x_1,y_1), \pi(x_2,y_2)) < \varepsilon$$

$$d(\pi(x_1,y_1), \pi(x_2,y_2)) = d((x_1,x_1),(x_2,x_2)) = \sqrt{(x_2-x_1)^2+(x_2-x_1)^2} =$$
$$= \sqrt{2} \cdot |x_2-x_1|$$

$|x_2-x_1| \leq \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} < \delta \quad \Rightarrow \sqrt{2} \cdot |x_2-x_1| < \sqrt{2}\,\delta = \varepsilon$

Stavimo $\delta := \frac{\varepsilon}{\sqrt{2}}$. Tada $\forall (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ vrijedi

ako $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} < \frac{\varepsilon}{\sqrt{2}} \Rightarrow |x_2-x_1| \leq \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} < \frac{\varepsilon}{2}$

$$\Rightarrow \sqrt{2} \cdot |x_2-x_1| < \varepsilon$$

$$\Rightarrow d(\pi(x_1,y_1), \pi(x_2,y_2)) < \varepsilon$$

$\Rightarrow$ π je uniformno neprekidna na $\mathbb{R}^2$.

2. popravak PPZ 2020 4b) Neka je funkcija $f:\mathbb{R}^n \to \mathbb{R}$ derivabilna na $\mathbb{R}^n$ i $g:\mathbb{R}\to\mathbb{R}^n$ derivabilna na $\mathbb{R}$. Zapišite sve parcijalne derivacije funkcija $f\circ g$ i $g\circ f$ preko parcijalnih derivacija funkcija f i g

Rj: $f\circ g:\mathbb{R}\to\mathbb{R}$, $g\circ f:\mathbb{R}^n\to\mathbb{R}^n$

Koristimo lančano pravilo.

$$(f\circ g)'(t)=\sum_{k=1}^{n}\frac{\partial f}{\partial x_k}(g(t))\cdot g_k'(t)$$

$$f'(g(t))=\left[\frac{\partial f}{\partial x_1}(g(t))\ \frac{\partial f}{\partial x_2}(g(t))\ \dots\ \frac{\partial f}{\partial x_n}(g(t))\right]$$

$$g'(t)=\begin{bmatrix} g_1'(t)\\ g_2'(t)\\ \vdots\\ g_n'(t)\end{bmatrix}$$

$$f'(g(t))\cdot g'(t)=\left[\frac{\partial f}{\partial x_1}(g(t))\ \frac{\partial f}{\partial x_2}(g(t))\ \dots\ \frac{\partial f}{\partial x_n}(g(t))\right]\cdot\begin{bmatrix} g_1'(t)\\ \vdots\\ g_n'(t)\end{bmatrix}$$

$$(g\circ f)'(x) = g'(f(x))\cdot f'(x) =$$

$$= \begin{bmatrix} g_1'(f(x)) \\ g_2'(f(x)) \\ \vdots \\ g_n'(f(x)) \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) & \frac{\partial f}{\partial x_2}(x) & \cdots & \frac{\partial f}{\partial x_n}(x) \end{bmatrix}$$

$$= \begin{bmatrix} g_1'(f(x))\cdot\frac{\partial f}{\partial x_1}(x) & g_1'(f(x))\cdot\frac{\partial f}{\partial x_2}(x) & \cdots & g_1'(f(x))\frac{\partial f}{\partial x_n} \\ \vdots & & & \\ g_n'(f(x))\cdot\frac{\partial f}{\partial x_1}(x) & & \cdots & g_n'(f(x))\cdot\frac{\partial f}{\partial x_n}(x) \end{bmatrix}$$

$$\frac{\partial (g\circ f)_j}{\partial x_i}\underset{\in \mathbb{R}^n}{(x)} = g_j'(f(x))\cdot\frac{\partial f}{\partial x_i}(x)$$

pročitamo iz

j-tog retka, i-tog stupca

$\boxed{2.\ \text{kol.}\ 2021.\ \ 4b)}$

Odredi Taylorov polinom reda 3 funkcije $f(x,y) = \sin(3x-y)$ u točki $(0,0)$.

$$T_3(0,0) = f(0,0) + \frac{1}{1!}\left(\frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0)\right)$$

$$+ \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(0,0)(x-0)^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)(x-0)(y-0)\right.$$

$$\left. + \frac{\partial^2 f}{\partial y \partial x}(y-0)(x-0) + \frac{\partial^2 f}{\partial y^2}(0,0)(y-0)^2\right)$$

$$+ \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}(0,0)(x-0)^3 + \frac{\partial^3 f}{\partial x^2 \partial y}(0,0)(x-0)^2(y-0)\right.$$

$$+ \frac{\partial^3 f}{\partial x \partial y \partial x}(0,0)(x-0)(y-0)(x-0) + \frac{\partial^3 f}{\partial y \partial x^2}(0,0)(y-0)(x-0)^2$$

$$+ \frac{\partial^3 f}{\partial x \partial y^2}(0,0)(x-0)(y-0)^2 + \frac{\partial^3 f}{\partial y \partial x \partial y}(0,0)(y-0)(x-0)(y-0)$$

$$\left. + \frac{\partial^3 f}{\partial y^2 \partial x}(0,0)(y-0)^2(x-0) + \frac{\partial^3 f}{\partial y^3}(0,0)(y-0)^3\right)$$

$$\frac{\partial f}{\partial x}(x,y) = 3\cos(3x-y)$$

$$\frac{\partial f}{\partial y}(x,y) = -\cos(3x-y)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -9\sin(3x-y)$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -3\sin(3x-y)\cdot(-1)$$

$$= 3\sin(3x-y)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -\sin(3x-y)$$

$$\frac{\partial^3 f}{\partial x^3}(x,y) = -9\cos(3x-y)\cdot 3 = -27\cos(3x-y)$$

$$\frac{\partial^3 f}{\partial x \partial y^2}(x,y) = -\cos(3x-y)\cdot 3 = -3\cos(3x-y)$$

$$\frac{\partial^3 f}{\partial x^2 \partial y}(x,y) = \left(3\cos(3x-y)\right)\cdot 3 = 9\cos(3x-y)$$

$$\frac{\partial^3 f}{\partial y^3}(x,y) = -\cos(3x-y)\cdot(-1) = \cos(3x-y)$$

$$T_3(0,0) = \frac{1}{1!}(3x-y) + \frac{1}{3!}\left(-27x^3 + 3\cdot 9x^2y + 3\cdot(-3)xy^2 + y^3\right)$$

$$= 3x - y + \frac{1}{6}\left(-27x^3 + 27x^2y - 9xy^2 + y^3\right)$$