Popravni 2020/2021

Zad 4. (a) Neka su $x, y \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$, $\varphi: \mathbb{R} \to \mathbb{R}^n$ funkcije zadane s

$$f(z_1, z_2, \dots, z_n) = z_1 z_n,$$
$$\varphi(t) = x + t(y-x).$$

Odredite $(f \circ \varphi)^{(2)}(t)$.

rj: $f \circ \varphi : \mathbb{R} \to \mathbb{R}$, $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$

$$(f \circ \varphi)(t) = f(\varphi(t)) = f(x + t(y-x)) =$$
$$= f((x_1, x_2, \dots, x_n) + t((y_1, y_2, \dots, y_n) - (x_1, x_2, \dots, x_n)))$$
$$= f(x_1 + t(y_1 - x_1), x_2 + t(y_2 - x_2), \dots, x_n + t(y_n - x_n))$$
$$= (x_1 + t(y_1 - x_1))(x_n + t(y_n - x_n)) =$$
$$= x_1 x_n + t((y_1 - x_1)x_n + x_1(y_n - x_n))$$
$$+ t^2 (y_1 - x_1)(y_n - x_n)$$

$$(f \circ \varphi)'(t) = (y_1 - x_1)x_n + x_1(y_n - x_n) + 2t(y_1 - x_1)(y_n - x_n)$$

$$(f \circ \varphi)^{(2)}(t) = 2(y_1 - x_1)(y_n - x_n)$$

(b) Neka su (X, d) i (Y, ρ) metrički prostori, $f: X \to Y$ neprekidna funkcija i $P \subset X$ putevima povezan skup u X. Dokažite da je tada $f(P)$ putevima povezan skup u Y.

<u>dokaz</u>: Neka su y_0 i $y_1 \in f(P)$. Tada postoje $x_0, x_1 \in P$ t.d. $y_0 = f(x_0)$, $y_1 = f(x_1)$.

Kako je P putevima povezan, točke $x_0, x_1 \in P$ se mogu spojiti putem, tj. postoji $\boxed{\text{neprekidno}}$ preslikavanje $\varphi : [0,1] \to X$ t.d. $\varphi(0) = x_0$, $\varphi(1) = x_1$ i $\varphi([0,1]) \subseteq P$.

Kako su f i φ neprekidne funkcije, onda je i $f \circ \varphi : [0,1] \to f(X)$ neprekidna funkcija i vrijedi:

$$(f \circ \varphi)(0) = f(\varphi(0)) = f(x_0) = y_0$$

$$(f \circ \varphi)(1) = f(\varphi(1)) = f(x_1) = y_1$$

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$$(f \circ \varphi)([0,1]) = f(\varphi([0,1])) \subseteq f(P)$$

$\Rightarrow$ $f \circ \varphi$ je put između y_0 i y_1

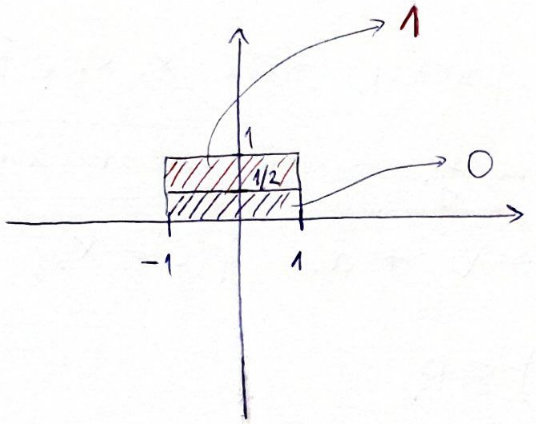
$\Rightarrow$ $f(P)$ je putevima povezan

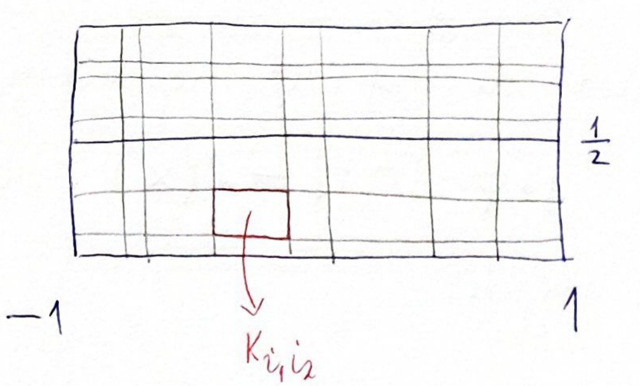
(c) Neka je $f:[-1,1]\times[0,1]\to\mathbb{R}$ funkcija zadana s

$$f(x,y)=\begin{cases}0, & y\le\frac{1}{2}\\ 1, & y>\frac{1}{2}\end{cases}$$

Je li funkcija f Riemann integrabilna na $[-1,1]\times[0,1]$? Obrazložite odgovor koristeći definiciju Riemannovog integrala.

Rj:





Neka je P proizvoljna subdivizija pravokutnika $K=[-1,1]\times[0,1]$.

- Ako je $K_{i_1 i_2}$ t.d. $y\le\frac{1}{2}$ $\forall$ $(x,y)\in K_{i_1 i_2}$, onda je

$$m_{i_1 i_2}=\inf\{f(x,y):(x,y)\in K_{i_1 i_2}\}=\inf\{0\}=0$$
$$M_{i_1 i_2}=\sup\{f(x,y):(x,y)\in K_{i_1 i_2}\}=\sup\{0\}=0$$

Ako je $K_{i_1 i_2}$ t.d. $y > \frac{1}{2}$ $\forall (x,y) \in K_{i_1 i_2}$, onda je

$$m_{i_1 i_2} = \inf\{f(x,y) : (x,y) \in K_{i_1 i_2}\} = \inf\{1\} = 1$$

$$M_{i_1 i_2} = \sup\{f(x,y) : (x,y) \in K_{i_1 i_2}\} = \sup\{1\} = 1$$

$$s(f,P) = \sum_{i_1=1}^{p_1} \sum_{i_2=1}^{p_2} m_{i_1 i_2} \nu(K_{i_1 i_2}) =$$

$$= \sum_{\substack{i_1=1 \\ y \le \frac{1}{2}\ \forall (x,y) \in K_{i_1 i_2}}}^{p_1} \sum_{i_2=1}^{p_2} \overset{0}{\overset{\parallel}{m_{i_1 i_2}}} \nu(K_{i_1 i_2}) + \sum_{\substack{i_1=1 \\ y > \frac{1}{2}\ \forall (x,y) \in K_{i_1 i_2}}}^{p_1} \sum_{i_2=1}^{p_2} \overset{1}{\overset{\parallel}{m_{i_1 i_2}}} \nu(K_{i_1 i_2})$$

$$= \sum_{\substack{i_1=1 \\ y > \frac{1}{2}\ \forall (x,y) \in K_{i_1 i_2}}}^{p_1} \sum_{i_2=1}^{p_2} \nu(K_{i_1 i_2}) = (1-(-1)) \cdot \left(\frac{1}{2}\right) = 1$$

$$S(f,P) = \sum_{\substack{i_1=1 \\ y \le \frac{1}{2}\ \forall (x,y) \in K_{i_1 i_2}}}^{p_1} \sum_{i_2=1}^{p_2} \overset{0}{\overset{\parallel}{M_{i_1 i_2}}} \nu(K_{i_1 i_2}) + \sum_{\substack{i_1=1 \\ y > \frac{1}{2}\ \forall (x,y) \in K_{i_1 i_2}}}^{p_1} \sum_{i_2=1}^{p_2} \overset{1}{\overset{\parallel}{M_{i_1 i_2}}} \nu(K_{i_1 i_2})$$

$$= (1-(-1)) \cdot \frac{1}{2} = 1$$

$I_*(f,K) = \sup\{1\} = 1$, $I^*(f,K) = \inf\{1\} = 1 \Rightarrow I_*(f,K) = I^*(f,K)$

$\Rightarrow f$ je Riemann integrabilna

Zad 3: c) Je li komplement skupa cijelih brojeva otvoren/zatvoren/kompaktan u $\mathbb{R}$? Svoje tvrdnje obrazložite.

rj: $S = \cancel{\mathbb{Z}}^c = \{\dots, -2, -1, 0, 1, 2, \dots\}^c =$

$= \dots \langle -3, -2\rangle \cup \langle -2, -1\rangle \cup \langle -1, 0\rangle \cup \langle 0, 1\rangle \cup \langle 1, 2\rangle \cup \dots$

$= \bigcup\limits_{m \in \mathbb{Z}} \underbrace{\langle m, m+1\rangle}_{\text{otvoren u } \mathbb{R}}$

- $S = \mathbb{Z}^c$ je otvoren u $\mathbb{R}$ jer je S unija otvorenih skupova.

- $S^c = (\mathbb{Z}^c)^c = \mathbb{Z}$ $\quad \mathbb{Z}$ nije otvoren u $\mathbb{R}$ jer npr. za $0 \in \mathbb{Z}$ vrijedi da $\langle -\varepsilon, \varepsilon\rangle \nsubseteq \mathbb{Z}$ ni za koji $\varepsilon > 0$. $\Rightarrow$ S nije zatvoren u $\mathbb{R}$

- Kompaktni skupovi u $\mathbb{R}$ su omeđeni i zatvoreni. S nije ni omeđen ni zatvoren, a onda nije ni kompaktan.

Zad 3: a) Neka su $A, B \subseteq \mathbb{R}^+$ odozdo ograničeni skupovi. Dokažite da je $\inf(AB) = \inf A \cdot \inf B$

Rj: $C := \{ab : a \in A, b \in B\} = AB$

- $\forall a \in A$ vrijedi $a \geq \inf A$
  $\forall b \in B$ vrijedi $b \geq \inf B$

$a \geq \inf A \ /\cdot b$ $\qquad a \cdot b \geq \inf A \cdot b \geq \inf A \cdot \inf B$

$b \in B \Rightarrow b \geq 0$

$\Rightarrow \forall\, c = ab \in C$ je $c \geq \inf A \cdot \inf B$

$\Rightarrow \inf A \cdot \inf B$ je donja meda skupa C.

- Kako je C odozdo ograničen $\Rightarrow$ C ima infimum
  $\boxed{\inf A \cdot \inf B \leq \inf C}$ (1)
  Za $c = ab$ vrijedi $ab \geq \inf C \ /: b \Rightarrow a \geq \frac{\inf C}{b}$ $\forall a \in A$

$\Rightarrow \frac{\inf C}{b}$ je donja meda skupa A $\Rightarrow \frac{\inf C}{b} \leq \inf A$

$\Rightarrow \inf C \leq b \cdot \inf A \Rightarrow$

1° Ako je $\inf A \neq 0$, onda je $\frac{\inf C}{\inf A} \leq b$ $\forall b \in B$

$\Rightarrow \frac{\inf C}{\inf A}$ je donja meda skupa B

$\Rightarrow \frac{\inf C}{\inf A} \leq \inf B \Rightarrow \boxed{\inf C \leq \inf A \inf B}$ (2)

2° Ako je $\inf A = 0 \Rightarrow \frac{\inf C}{b} \leq 0 \Rightarrow \inf C \leq 0$ $\Big\} \Rightarrow \inf$
Kako je $C \subseteq \mathbb{R}^+$, onda je ujedno i $\inf C \geq 0$

$\Rightarrow \overset{0}{\widetilde{\inf C}} = \overset{0}{\widetilde{\inf A}} \cdot \inf B$

Ako je $\inf A \neq 0$ (1) & (2) $\Rightarrow$ $\inf C = \inf A \cdot \inf B$

2. kolokvij 2022. 3.(b) Neka je $f: \mathbb{R}^2 \to \mathbb{R}$ klase C^2 i neka je $\gamma: \mathbb{R} \to \mathbb{R}^2$ zadana kao $\gamma(t) = (t^2, 1)$.
Definirajte parcijalnu derivaciju $\frac{\partial f}{\partial y}(x_0, y_0)$ funkcije f u točki (x_0, y_0). Ako je $g = f \circ \gamma$, odredite $g''(t)$.

Rj: $$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

$g = f \circ \gamma : \mathbb{R} \to \mathbb{R}$

Koristimo lančano pravilo $g'(t) = (f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t)$

$$g'(t) = \begin{bmatrix} \frac{\partial f}{\partial x}(\gamma(t)) & \frac{\partial f}{\partial y}(\gamma(t)) \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \gamma_1}{\partial t}(t) \\ \frac{\partial \gamma_2}{\partial t}(t) \end{bmatrix} \begin{array}{l} \to \gamma_1'(t) \\ \to \gamma_2'(t) \end{array}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x}(t^2, 1) & \frac{\partial f}{\partial y}(t^2, 1) \end{bmatrix} \cdot \begin{bmatrix} 2t \\ 0 \end{bmatrix} =$$

$$= 2t \cdot \frac{\partial f}{\partial x}(t^2, 1)$$

$$g''(t) = \left(2t \cdot \frac{\partial f}{\partial x}(t^2, 1)\right)' = (2t)' \cdot \frac{\partial f}{\partial x}(t^2, 1) + (2t) \cdot \left(\frac{\partial f}{\partial x}(t^2, 1)\right)'$$

$$= 2 \cdot \frac{\partial f}{\partial x}(t^2, 1) + (2t) \cdot \left(\frac{\partial f}{\partial x}(t^2, 1)\right)' \quad (*)$$

$$\left(\frac{\partial f}{\partial x}(t^2, 1)\right)' = \left(\frac{\partial f}{\partial x}(\gamma(t))\right)' = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(\gamma(t)) & \frac{\partial^2 f}{\partial y \partial x}(\gamma(t)) \end{bmatrix} \cdot \begin{bmatrix} 2t \\ 0 \end{bmatrix}$$

$$= 2t \cdot \frac{\partial^2 f}{\partial x^2}(\gamma(t))$$

$$\Rightarrow g''(t) = 2 \cdot \frac{\partial f}{\partial x}(t^2, 1) + 4t^2 \cdot \frac{\partial^2 f}{\partial x^2}(\gamma(t)) = 2 \cdot \frac{\partial f}{\partial x}(t^2, 1) + 4t^2 \frac{\partial^2 f}{\partial x^2}(t^2, 1)$$

Zad 3. c) Dokažite da je funkcija $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = |xy|$ diferencijabilna u $(0,0)$ ali nije diferencijabilna niti na jednom otvorenom krugu s centrom $(0,0)$.

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Rj. Uočimo da je $\lim\limits_{h \to 0} \dfrac{|f((0,0)+h) - f(0,0) - 0h|}{\|h\|} = 0$

$$\lim_{h \to 0} \frac{|f(h_1,h_2) - f(0,0)|}{\|h\|} = \lim_{h \to 0} \frac{|h_1 h_2|}{\sqrt{h_1^2+h_2^2}} = \lim_{h \to 0} \frac{|h_1|}{\sqrt{h_1^2+h_2^2}} \cdot |h_2| = 0$$

$$|h_1| \leq \sqrt{h_1^2+h_2^2}$$

$$\frac{|h_1|}{\sqrt{h_1^2+h_2^2}} \leq 1 \quad / \cdot |h_2|$$

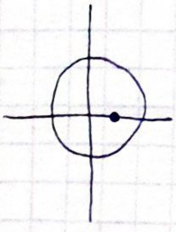
$$0 \leq \frac{|h_1 h_2|}{\sqrt{h_1^2+h_2^2}} \leq |h_2| \longrightarrow 0$$

$$\Rightarrow \lim_{h \to 0} \frac{|f(h_1,h_2) - f(0,0)|}{\|h\|} = 0$$

$\Rightarrow f$ je diferencijabilna u $(0,0)$

Ako bi f bila derivabilna (diferencijabilna) na nekom otvorenom krugu s centrom $(0,0)$, onda bi f imala na tom krugu sve parcijalne derivacije u svakoj točki tog kruga. Posebno, imala bi parcijalnu derivaciju po y u točki $(\varepsilon, 0)$ za neki $\varepsilon > 0$ (teorem 5.2. u skripti). Međutim

$$\cancel{\lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{|h|}{h}}$$



$$\lim_{h \to 0} \frac{f(\varepsilon,h) - f(\varepsilon,0)}{h} = \lim_{h \to 0} \frac{|\varepsilon h|}{h} = \lim_{h \to 0} \varepsilon \cdot \frac{|h|}{h}$$

Međutim $\lim\limits_{h\to 0^+} \varepsilon \cdot \frac{|h|}{h} = \varepsilon$
$\lim\limits_{h\to 0^-} \varepsilon \cdot \frac{|h|}{h} = -\varepsilon$ $\not=$ $\Rightarrow$ ne postoji parcijalna derivacija po y u točki $(\varepsilon, 0)$.