11 STRUJANJE PREKO ZVONOLIKE PREPREKE

• Problem: rješavamo diferencijalnu jednadžbu za 2D linearne uzgonske valove kada srednja osnovna struja naiđe na orografsku prepreku "zvonolikog" oblika ⇒ ovakav scenarij će generirati specifične uzgonske valove koji su vrlo bliski realnim slučajevima u atmosferi

- dif. jdžba za 2D lin. uzg. valove glasi:

$$\left(\cancel{\frac{\partial}{\partial t}}^{\,0} + \bar{u}\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right) + N^2\frac{\partial^2 w'}{\partial x^2} - \cancel{\frac{d^2\bar{u}}{dz^2}}^{\,0}\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial w'}{\partial x} = 0 \quad (*)$$

- ova općenita jdžba "dozvoljava" srednjoj struji ovisnost o z
- mi ovdje pojednostavljujemo stvari pa gledamo slučaj kada su i N i $\bar{u}$ konstante:

$$\Rightarrow \bar{u} = \text{const} \Rightarrow \frac{d^2\bar{u}}{dz^2} = 0$$

- planinski valovi su stacionarni pa komponenta operatora po vremenu iščezava $\Rightarrow \frac{\partial}{\partial t} \to 0$
- nakon ovih pojednostavljenja, (*) postaje:

$$\bar{u}^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right) + N^2\frac{\partial^2 w'}{\partial x^2} = 0 \quad \Big/ \cdot \frac{1}{\bar{u}^2} \text{ uz izlučivanje } \frac{\partial^2}{\partial x^2} \Rightarrow$$

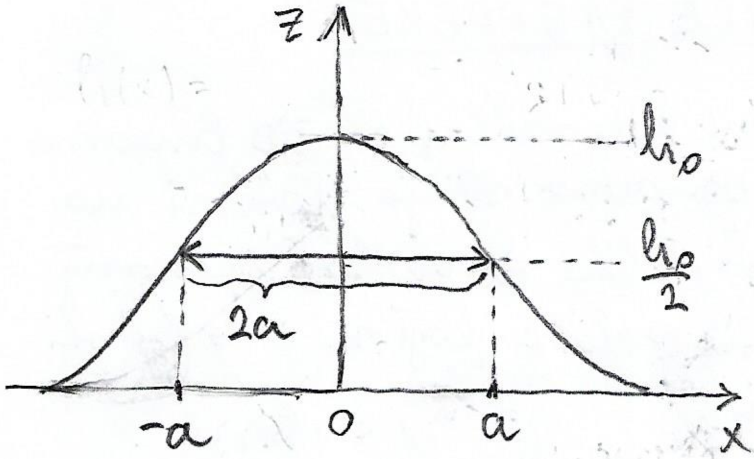
$$\Rightarrow \frac{\partial^2}{\partial x^2}\underbrace{\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}+\frac{N^2}{\bar{u}^2}w'\right)} = 0 \quad (**)$$

- digresija: $\frac{\partial^2}{\partial x^2}\left[f(x,z)\right] = 0$, ako je $f(x,z) \in \left[0, \text{const}, g(x)\right]$

⇒ dakle, argument u zagradi može biti ili 0 ili konstanta ili funkcija od x ⇒ kada bismo uzeli konstantu, stvar se dodatno komplicira, a kada bismo uzeli g(x), u priču dovodimo nekakvu vanjsku "silu prisile" o kojoj ništa ne znamo ⇒ dakle, uzimamo da je argument = 0!

$$\Rightarrow \frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}+\frac{N^2}{\bar{u}^2}w' = 0 \quad (***)$$

- sada uvodimo funkciju oblika zvonolike prepreke:



$$h(x) = h_0 \frac{a^2}{a^2 + x^2}$$

h_0 ... visina prepreke

a ... polušírina prepreke

- što je omjer $\frac{N^2}{\bar{u}}$ u (***)? ⇒ znamo da je Scorerov parametar oblika: $l^2 = \frac{N^2}{\bar{u}^2} - \cancel{\frac{1}{\bar{u}} \frac{d^2 \bar{u}}{dz^2}} \to 0$, jer u našem slučaju $\bar{u} = const$

⇒ dakle, omjer $\frac{N^2}{\bar{u}^2}$ u (***) je Scorerov parametar l^2

- sada u jdžbi (***) ispuštamo znak perturbacije ':

$$\Rightarrow \boxed{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} + l^2 w = 0} \quad (\bullet)$$

- u diferencijalnu jdžbu (•) ulazimo s probnim rješenjem u obliku inverznog Fourierovog transforma (IFT)
- općenito:

$$\text{FT} \ldots\ldots \; F(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx$$

$$\text{IFT} \ldots\ldots \; f(x) = \int_{-\infty}^{+\infty} F(k)\, e^{ikx}\, dk$$

- sada:

$$w(x,z) = \int_{-\infty}^{\infty} \tilde{w}(k,z)\, e^{ikx}\, dk \quad (\bullet\bullet)$$

- za fju $\tilde{w}(k,z)$ pretpostavimo da je oblika:

$$\tilde{w}(k,z) = \hat{w}(k)\, e^{imz}$$

$$\Rightarrow w(x,z) = \int_{-\infty}^{\infty} \hat{w}(k)\, e^{i(kx + mz)}\, dk$$

- trebaju nam rubni uvjeti (RU):

① donji rubni uvjet : $w(x, z=0) = \frac{dh}{dt} = \cancel{\frac{\partial h}{\partial t}} + \bar{u}\frac{\partial h}{\partial x}$

IFT ... $h(x) = \int_{-\infty}^{\infty} \tilde{h}(k) e^{ikx} dk$

$$\Rightarrow w(x, z=0) = \int_{-\infty}^{\infty} \tilde{w}(k, z=0) e^{ikx} dk = \bar{u}\, ik \int_{-\infty}^{\infty} \tilde{h}(k) e^{ikx} dk$$

$$\Rightarrow \boxed{\tilde{w}(k, z=0) = ik\bar{u}\,\tilde{h}(k)}$$

② gornji rubni uvjet : radijacioni rubni uvjet $\Rightarrow$ valni brojevi k i m su istog predznaka

- sada ćemo IFT od w (••) ubaciti u (•):

$$\Rightarrow (ik)^2 \int_{-\infty}^{\infty} \tilde{w}(k, z) e^{ikx} dk + \int_{-\infty}^{\infty} \frac{\partial^2 \tilde{w}(k, z)}{\partial z^2} e^{ikx} dk +$$

$$+ l^2 \int_{-\infty}^{\infty} \tilde{w}(k, z) e^{ikx} dk = 0 \Rightarrow$$

$$\Rightarrow \int_{-\infty}^{\infty} \underbrace{\left[-k^2 \tilde{w}(k, z) + \frac{\partial^2 \tilde{w}(k, z)}{\partial z^2} + l^2 \tilde{w}(k, z)\right]} e^{ikx} dk = 0$$

$$\Rightarrow \frac{\partial^2 \tilde{w}(k, z)}{\partial z^2} + (l^2 - k^2)\tilde{w}(k, z) = 0 \Rightarrow$$

$$\Rightarrow \underbrace{(im)^2}_{-m^2} \cancel{\hat{w}(k)} \cancel{e^{imz}} + (l^2 - k^2)\cancel{\hat{w}(k)} \cancel{e^{imz}} = 0 \Rightarrow$$

$$\Rightarrow \boxed{m^2 = l^2 - k^2} \quad (\blacktriangle)$$

odnos valnih brojeva i Scorer-ovog parametra

- općenito, u (▲) imamo 2 slučaja:

1) $k < l \Rightarrow m = \pm\sqrt{l^2 - k^2} \Rightarrow \tilde{w}(k, z) = \hat{w}(k) e^{\pm i\sqrt{l^2 - k^2}}$

2) $k > l \Rightarrow m = \pm i\sqrt{k^2 - l^2} \Rightarrow \tilde{w}(k, z) = \hat{w}(k) e^{\mp\sqrt{k^2 - l^2}}$

- nas će zanimati specijalni (ekstremni) slučajevi:

1sp) $k \ll l \Rightarrow \frac{1}{a} \ll \frac{N}{\bar{u}} \Rightarrow a \gg \frac{\bar{u}}{N}$

$\Rightarrow m \approx \pm l = \pm \frac{N}{\bar{u}} \Rightarrow$ HIDROSTATIČKI REŽIM

2sp) $k \gg l \Rightarrow \frac{1}{a} \gg \frac{N}{\bar{u}} \Rightarrow a \ll \frac{\bar{u}}{N}$

$\Rightarrow m \approx \pm i k = \pm i \frac{1}{a} \Rightarrow$ NEHIDROSTATIČKI REŽIM

- ovdje ćemo za vježbu raditi 2sp):

NEHIDROSTATIČKI REŽIM

$\Rightarrow m = \pm i k = i|k| \Rightarrow \tilde{w}(k,z) = \hat{w}(k)\, e^{imz} = \hat{w}(k)\, e^{-|k|z}$

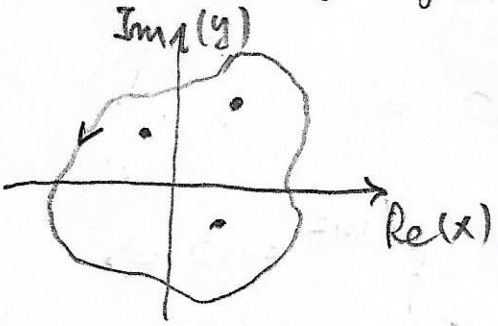
- sada nam treba $\hat{w}(k) \Rightarrow$ iz donjeg rubnog uvjeta, jer:

$\tilde{w}(k, z=0) = \hat{w}(k) = i k \bar{u}\, \tilde{h}(k) \Rightarrow$ kada izračunamo $\tilde{h}(k)$, u principu imamo sve...

- to ćemo izračunati koristeći FT: $\tilde{h}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x)\, e^{-ikx}\, dx$

$\Rightarrow \tilde{h}(k) = \frac{h_0 a^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2}\, dx$ (▲▲) ovaj integral rješavamo kompleksnom integracijom

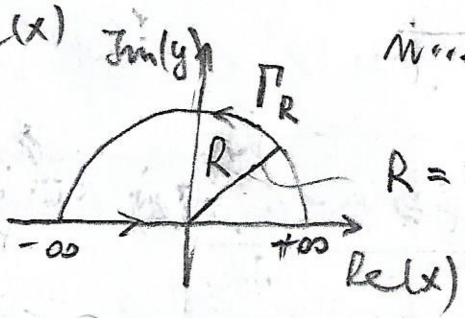
Kompleksna integracija



$$\frac{1}{2\pi i}\oint f(z)\, dz = \sum_{j=1}^{n} \text{Res}\, f(z)$$

n ... broj polova integrala

- Jordanova lema:



$R = \sqrt{x^2+y^2} = |z|$

$\Rightarrow$ ako $\lim_{|z|\to\infty} |f(z)| \to 0$, tada je $\int_{\Gamma_R} f(z)\, dz = 0$ pa se u tom slučaju rastavljeni integral $\oint$ pojednostavljuje $\Rightarrow$

$$\Rightarrow \frac{1}{2\pi i}\oint f(z)\,dz = \frac{1}{2\pi i}\left[\cancel{\int_{\Gamma_R} f(z)\,dz}^{\,0} + \int_{-\infty}^{\infty} f(x)\,dx\right] = \sum_{j=1}^{n} \operatorname{Res} f(z) \quad (\blacksquare)$$

- <u>Teorem o Residuumima:</u>

$$\operatorname*{Res}_{a_j} f(z) = \lim_{z\to a_j} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a_j)^m f(z)\right]$$

m... red j-tog pola

- sada, u našem slučaju $f(z) = \dfrac{e^{-ikz}}{z^2+a^2}$

- Jordanova lema: $\displaystyle\lim_{|z|\to\infty} |f(z)| = \lim_{|z|\to\infty}\left|\frac{e^{-ikz}}{z^2+a^2}\right| = \lim_{|z|\to\infty}\frac{e^{ikz}}{z^2+a^2}$

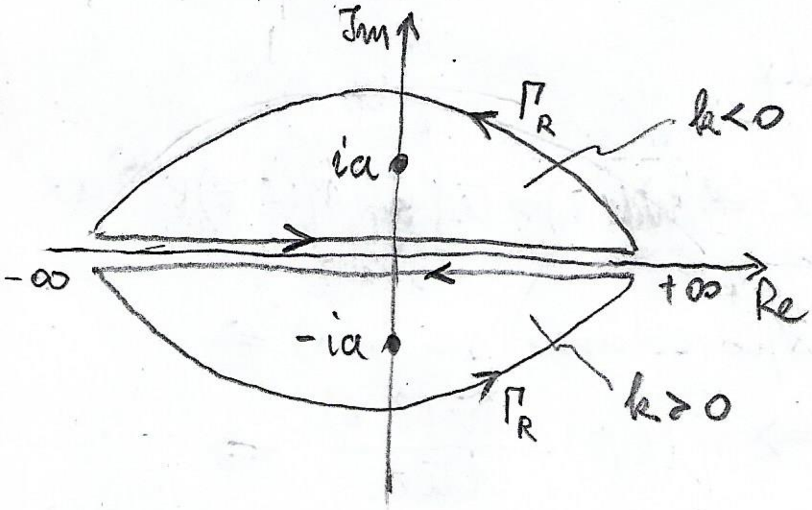
- gledamo u kojim uvjetima će J. lema biti zadovoljena:

(1) $\underbrace{z>0}_{\text{GORNJA POLURAVNINA}} \Rightarrow \displaystyle\lim_{|z|\to\infty}\frac{e^{ikz}}{z^2+a^2} \longrightarrow 0$ ako je $k<0 \Rightarrow \displaystyle\lim_{|z|\to\infty}\frac{e^{-i|k|z}}{z^2+a^2} = 0$

(2) $\underbrace{z<0}_{\text{DONJA POLURAVNINA}} \Rightarrow \displaystyle\lim_{|z|\to\infty}\frac{e^{-ik|z|}}{z^2+a^2} = 0$ ako je $k>0$

- naša fja $f(z)$ ima 2 pola 1. reda $(m=1)$:

$$f(z) = \frac{e^{-ikz}}{(z-ia)(z+ia)} \Rightarrow \text{polovi: } z = \pm ia$$



- $(\blacksquare)$ u gornjoj poluravnini:

$$\frac{1}{2\pi i}\oint f(z)\,dz = \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(x)\,dx$$

- $(\blacksquare)$ u donjoj poluravnini:

$$\frac{1}{2\pi i}\oint f(z)\,dz = \frac{1}{2\pi i}\int_{\infty}^{-\infty} f(x)\,dx$$

- gornja poluravnina $(k<0)$: $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \operatorname*{Res}_{z\to ia} f(z)$

- donja poluravnina $(k>0)$: $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = -\int_{\infty}^{-\infty} f(x)\,dx = -2\pi i \operatorname*{Res}_{z\to -ia} f(z)$

- sada rješavamo pojedinačne reziduume:

$$\operatorname*{Res}_{z\to ia} f(z) = \lim_{z\to ia}\left[\cancel{(z-ia)}\frac{e^{-ikz}}{\cancel{(z-ia)}(z+ia)}\right] = \frac{e^{ka}}{2ia}$$

$$\operatorname*{Res}_{z\to -ia} f(z) = \lim_{z\to -ia}\left[\cancel{(z+ia)}\frac{e^{-ikz}}{(z-ia)\cancel{(z+ia)}}\right] = \frac{e^{-ka}}{-2ia}$$

- sada imamo sve elemente za jdžbu (▲▲):

objedinjeno u $-|k|$

$$\Rightarrow \tilde{h}(k) = \frac{h_0 a^{\cancel{2}}}{2\pi}\begin{cases} \cancel{2\pi i}\,\dfrac{e^{ka}}{\cancel{2ia}}, & k<0 \\ \cancel{-2\pi i}\,\dfrac{e^{-ka}}{\cancel{-2ia}}, & k>0 \end{cases} = \frac{h_0 a}{2} e^{-|k|a}$$

- sada: $\hat{w}(k) = ik\bar{u}\,\frac{h_0 a}{2} e^{-|k|a} \Rightarrow$

$$\Rightarrow \tilde{w}(k,z) = ik\bar{u}\,\frac{h_0 a}{2} e^{-|k|a} e^{-|k|z}$$

- sada u priču uvedemo pojam strujnica $\eta(x,z)$ koje se pri tlu prilagođavaju terenu, a s $w(x,z)$ su povezane na sljedeći način: $w(x,z) = \bar{u}\,\frac{\partial \eta(x,z)}{\partial x}$

→ DONJI RU:

$$\Rightarrow w(x,z=0) = \bar{u}\,\frac{\partial \eta(x,z=0)}{\partial x} = \bar{u}\,\frac{\partial h(x)}{\partial x} \Rightarrow \eta(x,z=0) = h(x)$$

⇒ dakle, najdonja strujnica koincidira s terenom!

- FT: $\tilde{\eta}(k,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta(x,z) e^{-ikx}\,dx$

- IFT: $\eta(x,z) = \int_{-\infty}^{\infty} \tilde{\eta}(k,z) e^{ikx}\,dx \Rightarrow w(x,z) = \bar{u}\int_{-\infty}^{\infty} ik\,\tilde{\eta}(k,z) e^{ikx}\,dx$

- kada ovaj izraz za $w(x,z)$ usporedimo sa (●●) ⇒

$$\Rightarrow \tilde{w}(k,z) = i\bar{u}k\,\tilde{\eta}(k,z) \Rightarrow \tilde{\eta}(k,z) = \frac{\tilde{w}(k,z)}{i\bar{u}k} \Rightarrow$$

$$\Rightarrow \tilde{\eta}(k,z) = \frac{h_0 a}{2} e^{-|k|a} e^{-|k|z}$$

- kada izračunamo $\eta(x,z)$, gotovi smo jer iz nje prema definiciji dobijemo $w(x,z)$

- soda: $$\eta(x,z) = \frac{h_0 a}{2} \underbrace{\int_{-\infty}^{\infty} e^{-|k|a} e^{-|k|z} e^{ikx} dk}_{I}$$

I ⇒ rastavljamo na 2 integrala:

$$I = \underbrace{\int_{-\infty}^{0} e^{ka} e^{kz} e^{ikx} dk}_{k \to -k} + \int_{0}^{\infty} e^{-ka} e^{-kz} e^{ikx} dk =$$

$$= \int_{\infty}^{0} e^{-ka} e^{-kz} e^{-ikx} d(-k) + \int_{0}^{\infty} e^{-ka} e^{-kz} e^{ikx} dk =$$

$$= \int_{0}^{\infty} e^{-(a+z+ix)k} dk + \int_{0}^{\infty} e^{-(a+z-ix)k} dk$$

- ovaj integral se može naći u Bronštejnu: $\int_0^\infty t^n e^{-\beta t} dt = \frac{\Gamma(n+1)}{\beta^{n+1}}$, za $n > -1$ ⇒ u našem slučaju $n = 0$

$$\Rightarrow I = \frac{\Gamma(1)}{a+z+ix} + \frac{\Gamma(1)}{a+z-ix} = \frac{2(a+z)}{(a+z)^2 + x^2}$$

$$\Rightarrow \eta(x,z) = h_0 \frac{a(a+z)}{(a+z)^2 + x^2}$$

- provjera: $\eta(x, z=0) = h_0 \frac{a^2}{a^2+x^2} = h(x)$ ✓ OK

DZ: izračunati $w(x,z)$ i $u(x,z)$ te nacrtati $\eta(x,z)$ po visini

- hint za crtanje: znamo da je $\eta(x, z=0) = h(x)$ te da je $\lim_{z\to\infty} \eta(x,z) = 0$ ...

- ZA ONE KOJI ŽELE VIŠE: riješiti hidrostatički režim ...