$\boxed{2019.\ \underline{2\ zad}}$ $A: \mathbb{R}^3 \longrightarrow \mathcal{P}_2$

$$A(x_1, x_2, x_3) = (-2x_1 - 3x_2 + 3x_3) + x_2 t + (-x_1 - x_2 + x_3)t^2$$

$$(e') = \{(-1,-1,0), (1,-1,0), (0,0,1)\}$$

$$A(f', e') = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$(f') = ?$$

rj: $A(f', e') = I(f', f)\, A(f, e)\, I(e, e')$, gdje je

$(e) = \{e_1, e_2, e_3\}$ kanonska baza za $\mathbb{R}^3$, a

$(f) = \{1, t, t^2\}$ baza za $\mathcal{P}_2$

$$A(f, e) = \begin{array}{c} \begin{bmatrix} -2 & -3 & 3 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \\ \begin{matrix} A(e_1) & A(e_2) & A(e_3) \end{matrix} \end{array}$$

$$I(e, e') = \begin{array}{c} \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ e_1' & e_2' & e_3' \end{matrix} \end{array}$$

$$A(f,e)\cdot I(e,e') = \begin{bmatrix} 5 & 1 & 3 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A(f',e') = I(f',f)\cdot \underbrace{\begin{bmatrix} 5 & 1 & 3 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{C}$$

Uočimo da je C regularna matrica jer je

$$\det C = \det A(f,e)\cdot \det I(e,e') = 1\cdot 2 = 2 \neq 0$$

$$\Rightarrow \underbrace{A(f',e')\cdot C^{-1}}_{\text{regularna}} = \underbrace{I(f',f)}_{\text{regularna}} \qquad \Rightarrow A(f',e') \text{ regularna}$$

$$\Rightarrow I(f,f') = I(f',f)^{-1} = \left(A(f',e')\cdot C^{-1}\right)^{-1} =$$

$$= C\cdot A(f',e')^{-1} =$$

$$= \begin{bmatrix} 5 & 1 & 3 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} =$$

$$= \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ f_1' & f_2' & f_3' \end{array} \end{array}$$

$$\Rightarrow f_1' = 1+t^2$$

$$f_2' = 1-t$$

$$f_3' = 1+t$$

Dakle, baza (f') za $\mathcal{P}_3$ za koju je $A(f', e')$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$ je jedinstvena.

$\boxed{2017.\ 1}$ $$M = \left\{ X \in M_2(\mathbb{R}) : \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} X - X \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = 0 \right\}$$

$$f \in M^0, \quad f\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = -1$$

$$f\left(\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\right) = ?$$

Rj:

$$M = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} : \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} - \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} : \begin{bmatrix} x_1 & x_2 \\ x_1 - x_3 & x_2 - x_4 \end{bmatrix} - \begin{bmatrix} x_1 + x_2 & -x_2 \\ x_3 + x_4 & -x_4 \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} : \begin{bmatrix} -x_2 & 2x_2 \\ x_1 - 2x_3 - x_4 & x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} : x_1 - 2x_3 - x_4 = 0,\ x_2 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 & 0 \\ x_3 & x_1 - 2x_3 \end{bmatrix} : x_1, x_3 \in \mathbb{R} \right\} = \left[\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \right\} \right]$$

$$f \in M^0 \Rightarrow f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 \quad \& \quad f\left(\begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}\right) = 0$$

$$f\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = -1$$

Uočimo: $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\Rightarrow f\left(\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + f\left(\begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}\right) + 2 \cdot f\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$$

$$= 0 + 0 + 2 \cdot (-1) = \underline{\underline{-2}}$$

2017. 2.

$A \in L(\mathcal{P}_2)$ regularan

$(e) = \{1, t, t^2\}$

$(e') = \{1 - t + t^2, 2 + t^2, -1 + 2t - t^2\}$

$$A(e', e) = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 2 & -3 \\ -1 & 6 & 0 \end{bmatrix}$$

$(e'') = \{-1, 1 + t^2, t - 2t^2\}$

$A^{-1}(e'', e) = ?$

rj: $A^{-1}(e'', e) \cdot A(e, e'') = (A^{-1} \circ A)(e'', e'') = I(e'', e'') = I$

$$\Rightarrow \quad A^{-1}(e'', e) = (A(e, e''))^{-1} \qquad (1)$$

$A(e, e'') = I(e, e') \cdot A(e', e) \cdot I(e, e'') =$

$$= \begin{array}{c} \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix} \\ e_1' \quad e_2' \quad e_3' \end{array} \cdot \begin{bmatrix} 0 & 2 & -1 \\ 1 & 2 & -3 \\ -1 & 6 & 0 \end{bmatrix} \cdot \begin{array}{c} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\ e_1'' \quad e_2'' \quad e_3'' \end{array}$$

$$= \begin{bmatrix} -3 & -4 & 14 \\ 2 & -1 & 8 \\ -2 & -2 & 6 \end{bmatrix} \qquad (2)$$

(1) & (2) $\Rightarrow$ $\boxed{A^{-1}(e'', e) = \begin{bmatrix} -5 & 2 & 9 \\ 14 & -5 & -26 \\ 3 & -1 & -\frac{11}{2} \end{bmatrix}}$

$\boxed{2013 \quad ②}$ $(e') = \{1-t, t, t^2+1\}$

$B : P_2 \to P_2$

$(Bp)(t) := p(-t) + p(0)t$

$$B(e'', e') = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$(e'') = ?$

rj:

$$B(e, e') = I(e, e'')\, B(e'', e')\, \underbrace{I(e', e')}_{I}$$

$$= I(e, e'')\, B(e'', e') \qquad (1) \qquad (e) = \{1, t, t^2\}$$

$$B(e, e') = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ B(e_1') \;\; B(e_2') \;\; B(e_3') \end{array}$$

$B(e_1')(t) = e_1'(-t) + e_1'(0)t = 1 + t + t = 1 + 2t$

$B(e_2')(t) = e_2'(-t) + e_2'(0)t = -t$

$B(e_3')(t) = e_3'(-t) + e_3'(0)t = (-t)^2 + 1 + t = 1 + t + t^2$

Uočimo, $\det B(e'', e') = \left(-\frac{1}{2}\right) \cdot \frac{1}{4} = -\frac{1}{8} \neq 0 \Rightarrow B(e'', e')$

je regularna. Iz (1) $\Rightarrow I(e, e'') = B(e, e') \cdot \left(B(e'', e')\right)^{-1}$

$$\Longrightarrow I(e,e'') = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 0 \\ -6 & 2 & -2 \\ -2 & 2 & 0 \end{bmatrix} = \begin{array}{ccc} \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 2 \\ -2 & 2 & 0 \end{bmatrix} \\ e_1'' \quad e_2'' \quad e_3'' \end{array}$$

$$\Longrightarrow e_1'' = -4e_1 - 2t^2, \quad e_2'' = 2e_1 + 2e_3, \quad e_3'' = 2e_2$$

$$\Longrightarrow e_1''(t) = -4 - 2t^2, \quad e_2''(t) = 2 + 2t^2, \quad e_3''(t) = 2t$$

$\boxed{2017 \ (3)}$ $\quad L : M_2(\mathbb{R}) \longrightarrow P_2$

$$L(A) = tr(A)t^2 + tr(AB)t + tr(BAB)$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

y: $L(\alpha A) = tr(\alpha A)t^2 + tr((\alpha A)B)t + tr(B(\alpha A)B) =$

$$= \alpha \cdot trA \cdot t^2 + tr(\alpha \cdot (AB))t + tr(\alpha \cdot (BAB)) =$$

$$= \alpha \cdot (trA \cdot t^2 + tr(AB)t + tr(BAB)) = \alpha \cdot L(A)$$

$$L(A_1 + A_2) = tr(A_1 + A_2)t^2 + tr((A_1 + A_2)B)t + tr(B(A_1 + A_2)B)$$

$$= (trA_1 + trA_2)t^2 + tr(A_1B + A_2B)t + tr(BA_1B + BA_2B$$

$$= (trA_1t^2 + tr(A_1B)t + tr(BA_1B))$$

$$+ (trA_2t^2 + tr(A_2B)t + tr(BA_2B)) = L(A_1) + L(A_2$$

$\Rightarrow$ L je linearni operator

$\ker A = ?$

$$L(A) = 0 \iff \operatorname{tr}A\, t^2 + \operatorname{tr}(AB)t + \operatorname{tr}(BAB) = 0_{\mathcal{P}_2} \iff$$

$$\operatorname{tr}A = 0,\ \operatorname{tr}(AB) = 0,\ \operatorname{tr}(BAB) = 0$$

$$A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \qquad \operatorname{tr}A = x_1 + x_4$$

$$AB = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x_2 & x_1 \\ x_4 & x_3 \end{bmatrix} \Rightarrow \operatorname{tr}(AB) = x_2 + x_3$$

$$BAB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 & x_1 \\ x_4 & x_3 \end{bmatrix} = \begin{bmatrix} x_4 & x_3 \\ x_2 & x_1 \end{bmatrix} \Rightarrow \operatorname{tr}(BAB) = x_1 + x_4$$

$$\Rightarrow x_1 + x_4 = 0,\ x_2 + x_3 = 0,\ x_1 + x_4 = 0$$

$$\Rightarrow \ker A = \left\{ \begin{bmatrix} x_1 & x_2 \\ -x_2 & -x_1 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\} =$$

$$= \Big[\underbrace{\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}}_{\text{baza za } \ker A} \Big] \Rightarrow d(A) = 2$$

$$\operatorname{Im} A = \{(x_1+x_4)t^2 + (x_2+x_3)t + (x_1+x_4) \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$$

$$= \{x_1 \cdot (t^2+1) + x_2 \cdot t + x_3 \cdot t + x_4 \cdot (t^2+1) \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$$

$$= [\underbrace{\{t^2+1, t\}}_{\text{baza za } \operatorname{Im} A}] \Rightarrow r(A) = 2$$

$\boxed{2013.}$ 3.

$\dim W < +\infty$

$V_1, V_2 \leq W$

$f \in (V_1 \cap V_2)^0$

Rj: $f \in (V_1 \cap V_2)^0 \Rightarrow f(x) = 0 \quad \forall x \in V_1 \cap V_2$

- Neka je $\{a_1, \dots, a_k\}$ baza za $V_1 \cap V_2$,
- nadopunimo je do baze $\{a_1, \dots, a_k, b_1, \dots, b_l\}$ za V_1 i $\{a_1, \dots, a_k, c_1, \dots, c_m\}$ za V_2. Sada je $\{a_1, \dots, a_k, b_1, \dots, b_l, c_1, \dots, c_m\}$ baza za $V_1 + V_2$.
- Nadopunimo je do baze $\{a_1, \dots, a_k, b_1, \dots, b_l, c_1, \dots, c_m, d_1, \dots, d_n\}$ za W.

Definiramo $f_1, f_2 \in W^*$ na sljedeći način:

$f_1(a_1) = \dots = f_1(a_k) = 0$

$f_1(b_1) = \dots = f_1(b_e) = 0$

$f_1(c_i) := f(c_i) \quad \forall\, i \in \{1, \dots, m\}$

$f_1(d_j) := f(d_j) \quad \forall\, j \in \{1, \dots, n\}$

f_1 prošinimo po linearnosti na W.

$f_2(a_1) = \dots = f_2(a_2) = 0$

$f_2(b_i) = -f(b_i) \quad \forall\, i \in \{1, \dots, e\}$

$f_2(c_1) = \dots = f_2(c_m) = 0$

$f_2(d_1) = \dots = f_2(d_n) = 0$

f_2 prošinimo po linearnosti na W.

Kako se f_1 poništava na bazi od $V_1 \Rightarrow f_1 \in V_1^0$.

—||— f_2 —||— $V_2 \Rightarrow f_2 \in V_2^0$.

Nadalje $f(a_i) = 0 = f_1(a_i) - f_2(a_i) \quad \forall\, i \in \{1, \dots, k\}$

$f(b_i) = \underbrace{f_1(b_i)}_{0} - \underbrace{f_2(b_i)}_{-f(b_i)} \quad \forall\, i \in \{1, \dots, e\}$

$f(c_i) = \underbrace{f_1(c_i)}_{f(c_i)} - \underbrace{f_2(c_i)}_{0} \quad \forall\, i \in \{1, \dots, m\}$

$f(d_i) = \underbrace{f_1(d_i)}_{f(d_i)} - \underbrace{f_2(d_i)}_{0} \quad \forall\, i \in \{1, \dots, m\}$

$\Rightarrow f = f_1 - f_2, \quad f_1 \in V_1^0, \ f_2 \in V_2^0$