## 2.1 Inertial frames in Newtonian spacetime

By the early years of the 20th century, the notion of inertial system seems to have been widely accepted as the basis for Newtonian mechanics. In writing “On the electrodynamics of moving bodies” in 1905, Einstein took it to be obvious to his readers that classical mechanics does not require a single privileged frame of reference, but an equivalence-class of frames, all in uniform motion relative to each other, and in any of which “the equations of mechanics hold good.” Two inertial frames with coordinates $(x, y, z, t)$ and $(x', y', z', t')$ are related by the *Galilean transformations*,

$$
\begin{gathered}
x' = x - vt \\
y' = y \\
z' = z \\
t' = t
\end{gathered}
$$

(assuming that the x-axis is defined to be the direction of their relative motion). These transformations clearly preserve the invariant quantities of Newtonian mechanics, i.e. acceleration, force, and mass (and therefore time, length, and simultaneity). As far as Newtonian mechanics was concerned, then, the problem of absolute motion was completely solved; all that remained was to express the equivalence of inertial frames in a simpler geometrical structure.

The lack of a privileged spatial frame, combined with the obvious existence of privileged states of motion — paths defined as rectilinear in space and uniform with respect to time — suggests that the geometrical situation ought to be regarded from a four-dimensional *spatio-temporal* point of view. The structure defined by the class of inertial frames can be captured in the statement that *space-time* is a four-dimensional affine space, whose straight lines (geodesics) *are the trajectories of particles in uniform rectilinear motion*. See Figure 4.

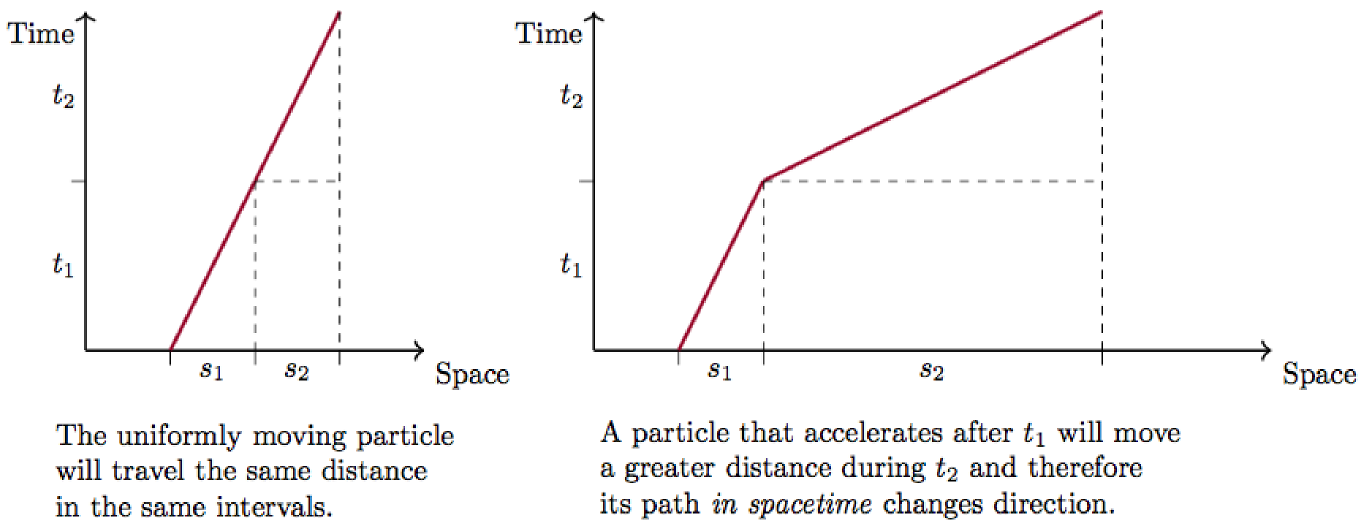



Figure 4: Inertial Trajectories as Straight Lines of Space-time

That is, space-time is a structure whose automorphisms — the Galilean transformations that relate one inertial frame to another — are affine transformations: they take straight lines into straight lines, and parallel lines into parallel lines. The former condition implies that an inertial motion in one frame will be an inertial motion in any other frame, and likewise for an accelerating or rotational motion. The latter implies that uniformly-moving particles or observers who are relatively at rest in one frame will also be relatively at rest in another. An inertial frame can be characterized as a family of parallel straight lines "filling" space-time, representing the possible trajectories of a family of free particles that are relatively at rest. See Figure 5. Therefore, to assert that an inertial frame exists is to impose a global structure on space-time; it is equivalent to the assertion that space-time is an affine space that is flat.

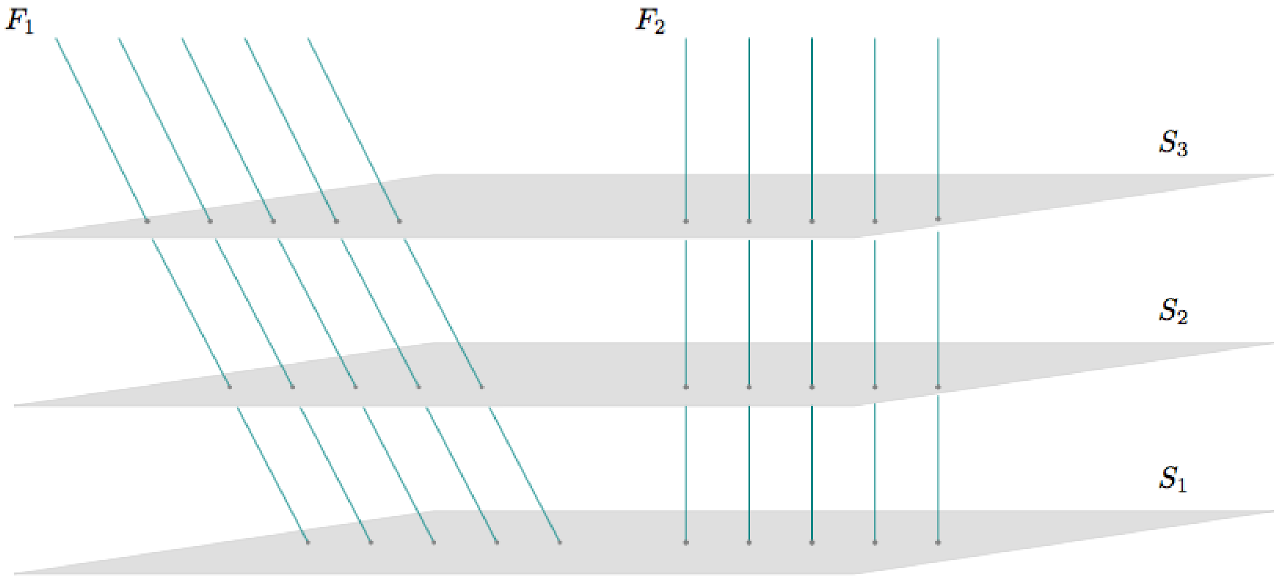
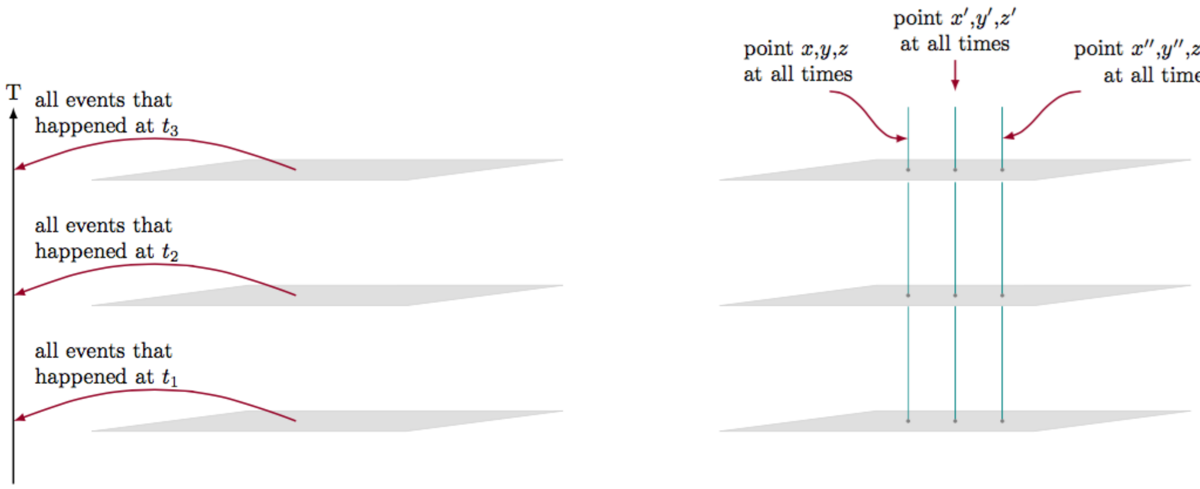



Figure 5: Each of these families of straight lines, *F1* and *F2*, represents the trajectories of a family of free particles that are relatively at rest, and therefore each defines an inertial frame. Relative to each other, the frames defined by *F1* and *F2* are in uniform motion. Each of the surfaces *S* is a "hypersurface of absolute simultaneity" representing all of space at a given moment; evidently (given the Galilean transformations) two inertial frames will agree on which events in space-time are simultaneous.

The form of the Galilean transformations shows that, in addition to being affine transformations, they also preserve metrical relations on time and space. Distinct inertial frames will agree on simultaneity, and on (ratios of) time-intervals; they will also agree on the spatial distance between points at a given moment of time. Therefore, in the four-dimensional picture, the decomposition of space-time into hypersurfaces of absolute simultaneity is independent of the choice of inertial frame. Another way of putting this is that Newtonian space-time is endowed with a *projection* of space onto time, i.e. a function that identifies space-time points that have the same time-coordinate. Similarly, absolute space arises from a projection of space-time onto space, i.e. a function that identifies space-time points that have the same spatial coordinates. See Figure 6.




The relation of simultaneity "decomposes" spacetime into 3-dimensional pieces, each representing "all of space at a given time," by projecting spacetime onto time, i.e., by identifying spacetime points that have the same time coordinates.

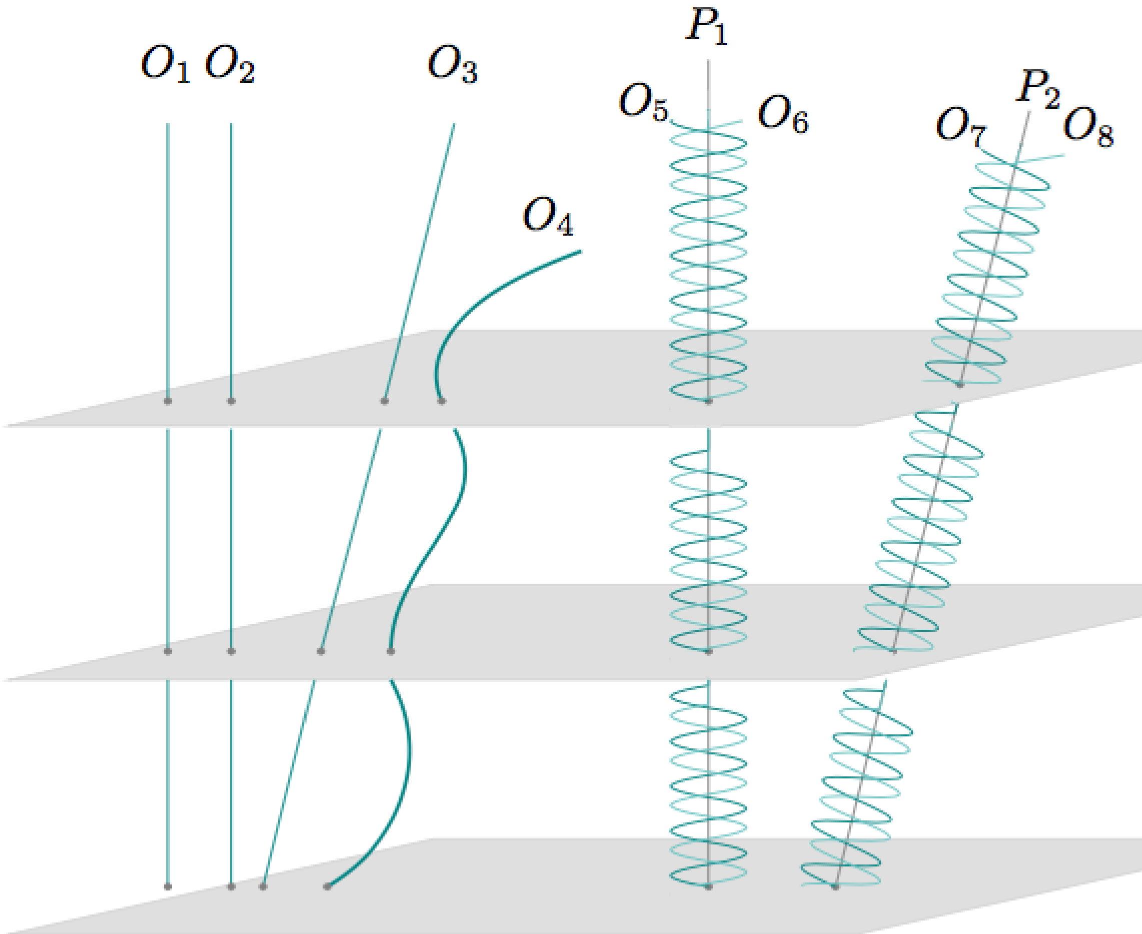
Similarly, one can think of the notion of "same place" as projecting spacetime onto space, i.e., by identifying spacetime points that have the same spatial coordinates; each of the trajectories thus singled out represents "a given place at all times.

Figure 6

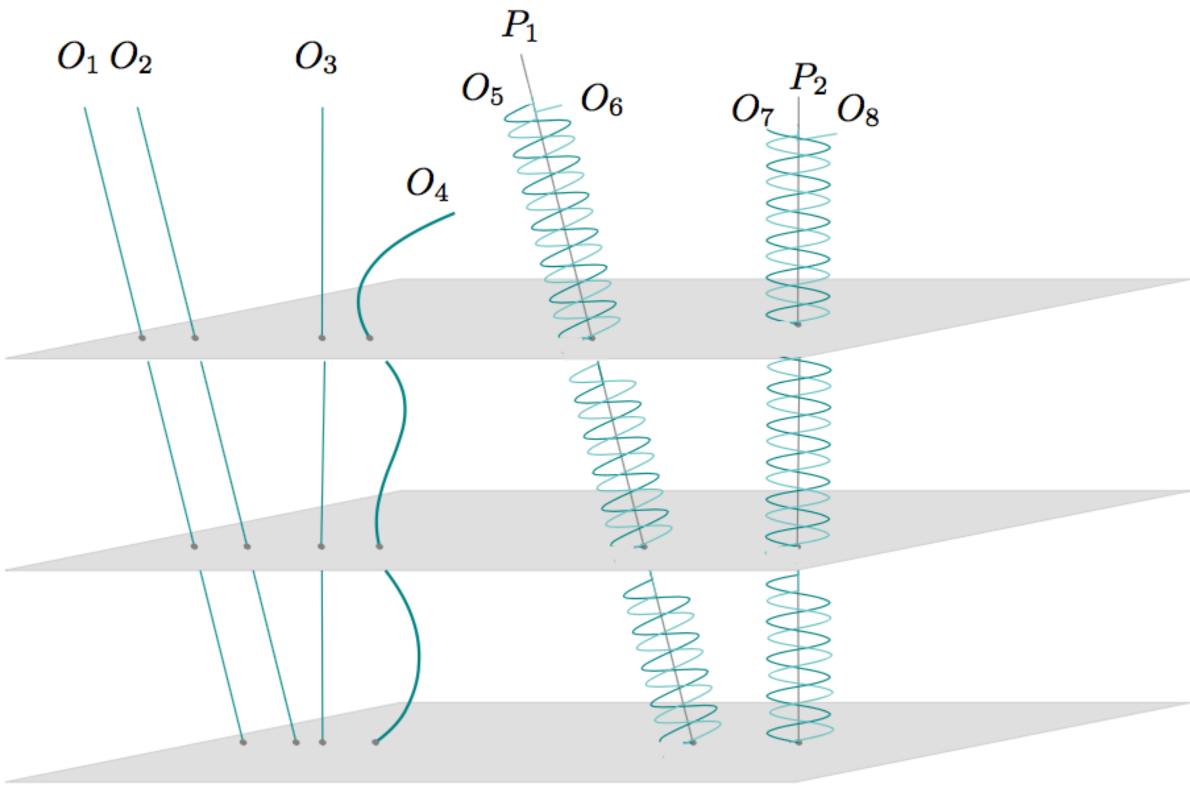
But Galilean relativity implies that this latter projection is arbitrary. While it assumes that we can identify the same time at different spatial locations, Newtonian mechanics provides no physical way of identifying the same spatial point at different times. Thus the equivalence of inertial frames can be thought of as the arbitrariness of the projection of space-time onto space. Any such projection is, essentially, the arbitrary choice of some particular inertial frame as a rest-frame. In the relativized version of Newton's theory, then, the class of inertial frames replaces absolute space, while absolute time remains. The structure of Newtonian space-time (also known as Galilean space-time, or neo-Newtonian space-time) expresses this fact in a direct and obvious way).

Figure 7: (a) Here is a space-time diagram of motions relative to the inertial frame in which $O_1$, $O_2$, and $P_1$ are at rest. This can be seen as arising from the projection of each of their inertial trajectories onto a single point of space. $O_3$ is in uniform motion. $O_4$ is accelerating any old way. $O_5$ and $O_6$ are revolving around their common center of gravity $P_1$, which (as noted above) is at rest. $O_7$ and $O_8$ are revolving around their center of gravity $P_2$, which is in uniform motion.

(b) Here is the same situation viewed from an inertial frame in which $O_3$ and $P_2$ are at rest. Now $O_1$, $O_2$, and $P_1$ are in uniform motion. $O_4$ is accelerating any old way. $O_5$ and $O_6$ are revolving around their common center of gravity $P_1$, which is in uniform motion. $O_7$ and $O_8$ are revolving around their center of gravity $P_2$, which (as noted above) is at rest.




(a)




(b)